1- Following the steps given in the lecture note, the state transformation z will be

$z_1 = x_{1,} \quad z_2 = x_{2,} \quad z_3 = g - \frac{k}{m}\, x_2 - \frac{L_0\, a\, x_3^2}{2\, m(a+x_1)^2}$

Applying control signal u=[(v+$\frac{k}{m}\, \dot{x_2} - \frac{L_0\, a\, x_3^2\, \dot{x_1}}{m(a+x_1)^3}) \frac{-m(a+x_1)^2}{L_0\; a\, x_3^3}]L + Rx_3 - \frac{L_0\, a\, x_2 x_3}{(a+x_1)^2}$

yields $\dot{z_1} = z_{2,} \quad \dot{z_2} = z_{3,} \quad \dot{z_3} = v$

where v=-$k_1\ddot{x}_1 - k_2\dot{x}_1 - k_3(x_1 - 0.05)$

k_i are found s.t make the above dynamics stable.

b) after taking time derivative of y for three times, u appears $\rightarrow r = n = 3$

There is no internal dynamics and the system is I/O linearizable

c) u is defined similar to a) but v is modified as follows:

v=$r^{(3)} - k_1\ddot{e} \quad - k_2\dot{e} \quad - k_3 e$, where e=y-r

and k_i are found s.t make the above dynamics stable.

2-$y = x_1 \rightarrow \dot{y} = x_2 + 2x_1^2 \rightarrow \ddot{y} = x_3 + u + 2x_1(x_2 + x_1^2)$

Relative degree: 2,

$$\mu_1 = x_1, \quad \mu_2 = x_2 + 2x_1^2$$

The third function ψ(x) is obtained by

$$L_g\, \psi = \frac{\partial \psi}{\partial x_2} = 0$$

Let us define $\dot{\psi} = \dot{x}_3 = x_1 - x_3$

Consider the Jacobean of state transformation z=[μ_1, μ_2, ψ].

$\begin{bmatrix} 1 & 4x_1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, it is full rank so the transformation is diffeomorphism.

Zero dynamics: $\dot{x}_3 = \ -x_3$, so the system is minimum phase.

Using Theorem of Lecture 8, page 71:

u=-$x_3 - 2\,x_1\,(x_2 + x_1^2) - \sin t - k_1\,\widetilde{\mu_1} - k_2\widetilde{\mu_2}$

k_1, k_2 are positive const. which make K(s)=$s^2 + k_2 s + k_1$ Hirwitz

3-

It is already in controllable canonical form:

A=$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\alpha(x) = \frac{a \sin x_1}{b \cos x_1}$, $\beta(x) = -\frac{1}{b \cos x_1}$

It is I/O linearizable when $\cos x_1 \neq 0$

b) Take u= $= -\frac{1}{b \cos x_1}$(-$a \sin x_1 + v$)

to stabilize the system at $x_1 = \theta$, take

v=-$k_1(x_1 - \ \theta) - k_2 x_2, k_1 > 0, k_2 > 0$

So the closed loop system is

$\dot{x_1}$ =x_2

$\dot{x_2} = -$=-$k_1(x_1 - \ \theta) - k_2 x_2$

It is a.s at equ. Point $(\theta, 0)$ in domain of $|x_1| < \ \pi/2$. So it is not g.a.s.

4-

$$f(x)=\begin{bmatrix}-x_1+x_2-x_3\\ -x_1x_3-x_2\\ -x_1\end{bmatrix},\ g(x)=\begin{bmatrix}0\\1\\1\end{bmatrix}$$

a) check the controllability and involutivity

$$ad_f g=\begin{bmatrix}0\\1+x_1\\0\end{bmatrix}$$

$$ad_f{}^2 g=\begin{bmatrix}-1-x_1\\1+x_2-x_3\\0\end{bmatrix}$$

$[g, ad_f g, ad_f{}^2 g]=\begin{bmatrix}0 & 0 & -1-x_1\\ 1 & 1+x_1 & 1+x_2-x_3\\ 1 & 0 & 0\end{bmatrix}$ for $x_1 \neq 1$ is linear independent so the system is controllable

$[g, ad_f g]=\begin{bmatrix}0 & 0\\ 1 & 1+x_1\\ 1 & 0\end{bmatrix}$ is inovlutive so the system is feedback linearizable

b) To find the transformation:

$$\nabla z_1 g = 0 \Rightarrow \frac{\partial z_1}{\partial x_2} + \frac{\partial z_1}{\partial x_3} = 0$$

$$\nabla z_1 ad_f g = 0 \Rightarrow \frac{\partial z_1}{\partial x_2}(1 + x_1) = 0$$

$$\nabla z_1 ad_f{}^2 g = 1 \Rightarrow -\frac{\partial z_1}{\partial x_1}(1 + x_1) + \frac{\partial z_1}{\partial x_2}(1 + x_2 - x_3) = 1$$

$$x_1 \neq 1$$

Therefore,

Z=T(x)= $\begin{bmatrix} x_1 \\ -x_1 + x_2 - x_3 \\ 2x_1 - 2x_2 + x_3 - x_1 x_3 \end{bmatrix}$

And

$$\dot{z_1} = z_2$$

$$\dot{z_2} = z_3$$

$$\dot{z_3} = v$$

And u= $\alpha(x) + \beta(x)v$

$$\alpha(x) = -\frac{x_3^2 + x_1^2 - 3x_1x_3 - x_2x_3 - 3x_1 + 2x_2 - 2x_3}{1 + x_1}$$

$$\beta(x) = -\frac{1}{1 + x_1}$$

5-Let us start with $\dot{x_1} = x_2 + \theta x_1^2$

And x_2 as its input. Let e=$x_1 - a\sin t$. Therefore,

$$\dot{e} = x_2 + \theta x_1^2 - a\cos t$$

Take $x_2 = -k_1 e + a\cos t = \varphi_1(e, t)$, where $k_1 > 0$. Consider $V_1 = \frac{1}{2}e^2$

Hence, $\dot{V_1} = -k_1\, e^2 + \theta\, e\, x_1^2 \leq -k_1\, e^2 + 2|\, e||\, x_1\, |^2 \leq\; -k_1\, e^2 + 2\,|e|(|e| + a)^2 \leq -k_1\, e^2 + 2|e|^3 + 4|e|^2 + 2|e|$

Choosing k_1 sufficiently large make e ultimately bounded.

Let us proceed the second step of back stepping: $z_1 = e, z_2 = x_2 - \varphi_1(e,t)$:

$$\dot{z}_1 = -k_1 z_1 + \theta x_1^2 + z_2$$

$$\dot{z}_2 = x_3 + u + k_1(-k_1 z_1 + \theta x_1^2 + z_2) + a \sin t$$

Let $V = \frac{1}{2} z_1{}^2 + \frac{b}{2} z_2{}^2 \rightarrow \dot{V} = z_1(-k_1 z_1 + \theta x_1^2 + z_2) + b^2 z_2[x_3 + u + k_1(-k_1 z_1 + \theta x_1^2 + z_2) + a \sin t]$

Therefore take u= - $[x_3 + k_1(-k_1 z_1 + z_2) + a \sin t] - \frac{1}{b^2}(k_2 z_2 + z_1)$

$\rightarrow \dot{V} = -k_1 z_1^2 + \theta x_1^2 z_1 + \theta x_1^2 b^2 z_2 k_1 - k_2 z_2^2$
$\leq -k_1 z_1^2 - k_2 z_2^2 + 2|z_1|(|z_1| + a)^2 + 2b^2|z_2| k_1(|z_1| + a)^2 \leq -k_1 z_1^2 - k_2 z_2^2 + 2|z_1|^3 + 4|z_1|^2 + 2|z_1| + 2b^2|z_2| k_1(|z_1| + 1)^2$

Due to the cubic terms on the right hand side, we should limit our analysis to compact set. Let $\Omega = (V \leq c)$.

Choose c>0 s.t. z(0) ϵ Ω. Using the fact that $\|x\|_\infty \leq 1 \rightarrow |z_1(0)| \leq 1$, $|z_2(0)| = |x_2(0) + k_1 e(0) - a \leq 2 + k1$

The initial state $x_2(0)$ depends on k_1. To choose c independent of k_1, take b=1/(2+k_1).

Then V(z(0))=1/2 $z_1^2(0) + \frac{1}{2(2+k_1)^2} z_2^2(0) \leq 1$

Therefore we take c=1 and set $\Omega = (V \leq 1) \rightarrow |z_1| \leq \sqrt{2}$

$$2|z_1|^3 + 4|z_1|^2 + 2|z_1| \leq (2\sqrt{2} + 4)|z_1|^2 + 2|z_1|$$

$$2b^2|z_2| k_1(|z_1| + 1)^2 \leq (2 + \sqrt{2})|z_1||z_2| + |z_2|$$

Take $k_1 = 2\sqrt{2} + 4 + 2\alpha$ and $k_2 = 2\alpha$ where $\alpha > 0$.

Hence, $\dot{V} \leq -\alpha \|z\|_2^2 + \sqrt{5}\|z\|_2 + \left[-\alpha\|z\|_2^2 + (2 + \sqrt{2})|z_1||z_2|\right]$

Choosing α large enough, one can achieve

$$\dot{V} \leq -\alpha \|z\|_2^2 + \sqrt{5}\|z\|_2 \leq -(1-\beta)\alpha \|z\|_2^2, \quad \forall\, \alpha \|z\|_2 \geq \frac{\sqrt{5}}{\alpha\beta}, \quad 0 < \beta < 1$$

Z is uniformly ultimately bounded and its bound is proportional to 1/ α.

Notice that in $\dot{x}_3$, the boundedness of x_1 implies the boundedness of x_3

6- a) Take u=-2$x_1 + x_1^3$-$x_2 = \phi(x)$, Therefore $\dot{x} = Ax$, where A=$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ that is Hurwitz. Hence the Orion is g.e.s.

b) Let $\xi = z - \phi(x)$

$$\dot{x} = Ax + B\xi$$

$$\dot{\xi} = v - \frac{\partial\phi}{\partial x}(Ax + B\xi)$$

Where $\begin{bmatrix} 0 \\ 1 \end{bmatrix}. Let\ V = x^T Px + \frac{1}{2}\xi^2 \rightarrow \dot{V} = -x^T x + 2x^T PB\xi + \xi\left[v - \frac{\partial\phi}{\partial x}(Ax + B\xi)\right]$

Therefore, take v=$\frac{\partial\phi}{\partial x}(Ax + B\xi) - 2x^T PB - \xi$, this makes the system g.a.s.

7- Divide both side of the system dynamics by b:

h $x^{(n)} = hf(x) + u$

The disturbance is additive so $|\hat{h} - h| < H$

s=$\tilde{x}^{n-1} + p(\tilde{x}^{n-2}, \dots, \tilde{x}) \rightarrow h\dot{s} = hf + u - hx_r{}^d - hq,$

where q= $\dot{p}$

$$\hat{u} = -\hat{h}\hat{f} + \hat{h}x_d^r + \hat{h}q$$

$$u = \hat{u} - ksgn(s)$$

h$\dot{s} = hf - \hat{h}\hat{f} + \hat{h}x_d^r - hx_d^r + \hat{h}q - ksgn(s)$-hq

To satisfy the mentioned condition h$\dot{s}s \leq -\eta\, h\,|s|$

k should satisfy: k $\geq |hf - \hat{h}\hat{f} + Hx_d^r + Hq| + \eta\, h$

Considering the known upper bound F and H the above condition is guaranteed if k $\geq ||\hat{h}|F + H(\hat{f} + F) + Hx_d^r + Hq| + \eta(|\hat{h}|+H)$

Boundary layer interpolations and time-variation of φ can be derived similarly to the standard case. Therefore, the accuracy of the approximate "bandwidth" analysis 10 the boundary layer increases with λ.

8- Define e=$\theta - \sigma \rightarrow \ddot{e} = \ddot{\theta},\ \dot{e} = \dot{\theta}$, therefore

$$\ddot{e} = -a\sin(e+\sigma) - b\dot{e} + cu$$

s=$\dot{e} + \lambda e$

let $\hat{a} = 5, |a - \hat{a}| < A = 10,\ \ \hat{b} = 0, \left|b - \hat{b}\right| < B = 0.2, \hat{c}$=$\sqrt{6 \times 12},\ \ \beta = \sqrt{\frac{12}{6}}$

Following the steps mentioned in the lecture note leads to:

$$\hat{u} = \hat{a}\ \sin(e+\sigma) - \hat{b}\dot{e}$$

$u = \hat{c}^{-1}[\hat{u} - k\ sgn(s)]$

After following some manipulations, one can easily get to the following condition for k

k$\geq \beta\ (\eta - A\ sin\ (e+\sigma) - B\ \dot{\theta}) + (1-\beta)|\hat{u}|$

Type equation here.