1- a)

$$0 = k_1 x_2\left(1 - \frac{x_1}{1+x_2^2}\right) \rightarrow x_2 = 0, \qquad or\ \ x_1 = 1 + {x_2}^2$$

$$x_2 = 0, 0 = k_2 - x_2 - \frac{4x_1x_2}{1+x_2^2} \ \rightarrow k_2 = 0\ contradicts\ wit\ assumption,$$

$$x_1 = 1 + {x_2}^2, 0 = k_2 - x_2 - \frac{4x_1x_2}{1+x_2^2} \quad \rightarrow \boldsymbol{EQU\ PT}: \quad x_2 = \frac{k_2}{5}, \quad x_1 = 1 + (\frac{k_2}{5})^2$$

(b)Linearization:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -\frac{k_1x_2}{1+x_2^2} & k_1 - \frac{k_1x_1}{1+x_2^2} + \frac{2k_1x_1x_2^2}{(1+x_2^2)^2} \\ \frac{-4x_2}{1+x_2^2} & -1 - \frac{4x_1}{1+x_2^2} + \frac{8x_1{x_2}^2}{(1+x_2^2)^2} \end{bmatrix}, \text{at equ pt: } \frac{x_1}{1+x_2^2} = 1 \ \rightarrow \frac{\partial f}{\partial x} = \frac{1}{1+x_2^2}\begin{bmatrix} -k_1x_2 & 2k_1x_2^2 \\ -4x_2 & -5 + 8{x_2}^2 \end{bmatrix}$$

Characteristic equation : $s^2 + s(5 - 8x_2^2 + k_1x_2) + 5k_1x_2 = 0$

At equ. Pt. : $s^2 + s\left(5 - 8(\frac{k_2}{5})^2 + \frac{k_1k_2}{5}\right) + k_1k_2 = 0$

Since Equ pt. is inside K, based on P-B we need to show the Equ. Pt. is unstable.

Re{λi}>0, since $k_1k_2 > 0$, so $5 - 8(\frac{k_2}{5})^2 + \frac{k_1k_2}{5} < 0 \rightarrow k_1 < 8(\frac{k_2}{5})^{2 -} \frac{25}{k_2}$

On the other hand, one should show the Region K is invariant, i.e. All trajectories form inside remains there.

On border of this region:

On$x_2 = k_2$: $\delta = x_2 \rightarrow \delta^2 = {x_2}^2 \rightarrow \frac{d\delta^2}{dt} = x_2\left(k_2 - x_2 - \frac{4x_1x_2}{1+x_2^2}\right) \xrightarrow{x_2 = k_2, x_1 > 0} -k_2^2\frac{4x_1}{1+k_2^2} < 0$

On $x_1 = 1 + k_2^2$: $\delta = 1 + x_2^2 \rightarrow \frac{d\delta^2}{dt} = k_1x_1x_2(1 - \frac{1+k_2^2}{1+x_2^2})$

For x_2>0, and $K_2 > x_2 \rightarrow \dot{\delta} < 0$

$x_1 = 0 \rightarrow \dot{x_1} = k_1x_1, x_2 \geq 0 \ \rightarrow \dot{x_1}\ \geq 0$

$x_2 = 0 \rightarrow \dot{x_2} = k_2 \geq 0$

$x_1 = 1 + k_2^2 \rightarrow \dot{x_1} = k_1x_2\left(1 - \frac{1+k_2^2}{1+x_2^2}\right) < 0\ if\ 0 < x_2 < K_2$

$x_2 = k_2 \rightarrow \dot{x_2} = -\frac{4x_1k_2}{1+k} < 0\ for\ x_1 > 0$

Therefore, K is invariant set

2- (a)

Let us define : y=$x_1, \dot{y} \; = x_2$

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -\mathrm{h}(x_1)x_2 - g(x_1)$$

To set the origin to be Equ. Pt.

$$\begin{matrix} 0 = x_2 \\ 0 = -\mathrm{h}(x_1)x_2 - g(x_1) \end{matrix} \rightarrow g(0) = 0$$

So: $g(0) = 0 \; and \; h(0) < M \; (h(0) should \; not \; be \; inifitniy$

(ii)

Consider the following Lyap. Fcn:

$$V = \frac{1}{2}x_2^2 + \int_0^{x_1} g(y)dy \rightarrow \dot{V} \; = x_2\, g(x_1) - x_2^2\, \mathrm{h}(x_1) - x_2\, g(x_1) = -x_2^2\, \mathrm{h}(x_1) \leq 0$$

$$\dot{V} = 0 \; \Rightarrow \begin{matrix} x_2 \equiv 0 \rightarrow \dot{x_1} = 0 \rightarrow x_1 = c, \dot{x_2} = 0 \rightarrow \; g(c) = 0 \rightarrow c = 0 \\ \mathrm{h}(x_1) \equiv 0 \rightarrow no \;\; result \end{matrix}$$

So based on Lassale sp a.s.: $\mathrm{h}(x_1) < 0$

3- Try $g(x) = \begin{bmatrix} \propto x_1 + \beta x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$,

To meet the symmetry requirement : $\gamma = \beta$

$\dot{V}(x) = (\propto x_1 + \beta x_2)x_2 - (\, \beta x_1 + \delta x_2)[(\, x_1 + x_2) + \sin\,(x_1 + x_2)]$

Take $\delta = \beta$

$$\dot{V}(x) = -\beta x_1^2 - (\alpha - 2\beta)x_1 x_2 - \beta(\, x_1 + x_2)\sin\,(x_1 + x_2)$$

Take $\alpha = 2\beta$ and $\beta > 0$:

$\dot{V}(x) = -\beta x_1^2 - \beta(\, x_1 + x_2)\sin\,(x_1 + x_2)$

Which is neg. def. for all xϵ R.

g(x)=$\beta \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x = Px \;\; \rightarrow V(x) = \; \int_0^x g^T(y)dy = \frac{1}{2}x^T Px$

P is pos. def. therefore, V(x) is radially unbounded ad origin is g.a.s.

4-origion is the only Equ. Point.

$$V(x) = x^T x = x_1^2 + x_2^2 + x_3^2$$

$\dot{V}(x) = 2[x_1x_2 - x_1x_2 - x_2^2 sat(x_2^2 - x_3^2)$+$x_3^2 sat(x_2^2 - x_3^2)]$=-2$(x_2^2 - x_3^2)sat(x_2^2 - x_3^2) \leq 0$

$\dot{V}(x) = 0 \rightarrow\ x_2^2(t) \equiv x_3^2(t) \rightarrow \dot{x}_3\dot{(t)} \equiv 0$

Therefore, $x_2(t) and\ x_3(t)$ are contestant $\rightarrow \dot{x}_2\dot{(t)} \equiv 0$

From the second equation $\rightarrow\ x_1\ (t) \equiv 0$

The first equation $\rightarrow\ x_2\ (t) \equiv 0$

Therefore $\rightarrow\ x_3\ (t) \equiv 0$

Using LaSall's theorem yield g.a.s of the equ point.

6- (3.6)

a) $\dot{x}$ =$f(t,x) \rightarrow x(t) = x_0(t) + \int_{t_0}^{t} f(\tau, x(\tau))d\tau\ \Rightarrow \|x(t)\| \leq \|x_0(t)\| + \int_{t_0}^{t}[k_1 + k_2\,\|x(\tau)\|]$ d$\tau = \|x_0(t)\| + k_1(t - t_0) + k_2 \int_{t_0}^{t}\|x(\tau)\|\ d\tau$

Using Grwonwall-Bellman inequality (ref. Appendix A.1):

Here $\lambda(t) = \|x_0(t)\| + k_1(t - t_0), \mu = k_2 \Rightarrow$

$$\|x(t)\| \leq \|x_0(t)\| + k_1(t - t_0) + \int_{t_0}^{t} [\|x_0(t)\| + k_1(s - t_0)]k_2 e^{k_2(t-s)}\ ds \Rightarrow$$

$$\|x(t)\| \leq \|x_0(t)\| e^{k_2(t-t_0)} + \frac{k_1}{k_2}\ [e^{k_2(t-t_0)}\quad - 1]$$

b) The upper bound of x(t) is exponential. Therefore, the upper bound is finite in finite t $\Rightarrow x(t)$ is finite in finite t $\Rightarrow$ There is no finite escape time.

5- f(x) is cont. diff. and also $\|f(x)\|\ \leq k_1$+$k_2\|x\|, \forall\ x \in R^2$. By applying the above exercise, one can conclude that there is no finite escape time and the unique existence of the solution is guaranteed globally.

7

a)

$$x_1 = 0 \Rightarrow V(x) = \frac{x_2^2}{1+x_2^2} + x_2^2 \rightarrow \infty \; as \; |x_2| \rightarrow \infty$$

$$x_2 = 0 \Rightarrow V(x) = \frac{x_1^2}{1+x_1^2} + x_1^2 \rightarrow \infty \; as \; |x_1| \rightarrow \infty$$

b) On the line $x_1 = x_2$, We have: $V(x) = \frac{4x_1^2}{1+4x_1^2} \rightarrow 1 \; as \; |x_1| \rightarrow \infty$

8- a) $$\begin{aligned} 0 &= x_2 \\ 0 &= -x_2 - \sin x_1 - 2sat(x_1 + x_2) \end{aligned}$$

So $\sin x_1 - 2sat(x_1) = 0 \rightarrow x_1$=0,

The origin is the only Equ. Point.

b) A=$\frac{\partial f}{\partial x}|_{x=0} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 - 2 & -1-2 \end{bmatrix}|_{x=0} = \begin{bmatrix} 0 & 1 \\ -3 & -3 \end{bmatrix}$

The Eigen values of A are -3/2$j\sqrt{3/2}$. So A is Hurwitz and origin is a.s.

c) $\sigma\dot{\sigma} = \sigma\,[-sin\, x_1 - 2\, sat(\sigma)] \leq |\sigma| - 2|\sigma| = -|\sigma|, \; for \; |\sigma| \geq 1$

d) $\dot{V}(x) = (2x_1 + \sin x_1)\, x_2 + x_2[-x_2 - \sin x_1 - 2\, sat(x_1 + x_2)] = -3x_2^2 \leq 0, \; for \; |\sigma| < 1$

M_c is closed and bounded by 4 parts: tow parts bounded by V(x)=c, one by line $\sigma = 1$

and the other one by line $\sigma = -1$

$\dot{V}(x) \leq 0 \; in$ Mc $\rightarrow$ trajectories cannot leave the set though the boundary provided by V(x)=c.

Using c) $\rightarrow$ trajectories cannot leave the set though the boundary provided by $\sigma = \pm 1$.

Therefore, every trajectory starting in Mc remain there for all future tiem.

Now $\dot{V}(x) = 0 \rightarrow x_2(t) \equiv 0 \rightarrow \sin(x_1(t)) - 2\, x_1(t) \equiv 0 \rightarrow x_1(t) \equiv 0$

Using the Lassal's theorem, one can conclude that every trajectory starting in Mc approaches the origin as $t \to \infty$

e) $\sigma\dot{\sigma} \leq -|\sigma| \ for \ |\sigma| \geq 1$ → every trajectory starting outside of the region $|\sigma| \leq 1$, must reach the origin in finite time.

For example, if the trajectory starts from the region $\sigma > 1 \to \sigma\dot{\sigma} \leq -\sigma \ \to \dot{\sigma} \leq -1 \ \to \ \sigma(t) \leq \sigma(0) - t$

It means that the traj. Reaches the region $|\sigma| \leq 1$ in time les than or equal $\sigma(0) - 1$

The similar story happens if the trajectory starts in the region $\sigma < -1$.

On the other hands, inside the region $|\sigma| \leq 1$ belongs to Mc which based on d) it is guaranteed that the traj. Reaches the origin as $t \to \infty$

Hence, the origin is g.a.s.

9- Consider the Lyapunov

$$V(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} (y - y^3)dy = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 - \frac{1}{4}x_1^4$$

V(x) is p.d in the region of $|x_1| < \sqrt{2}$

$$\dot{V}(x) = (x_1 - x_1^3)x_2 + x_2\,[-x_2 - (x_1 - x_1^3)] = -x_2^2 \leq 0$$

When $\dot{V}(x) = 0 \to x_2 \equiv 0 \ \to x_1 - x_1^3 \equiv 0 \ \to x_1 \ \equiv 0 \, for \ |x_1| < 1$

Therefore, for D={|x|<1} the origion is as. Now we should look for $\Omega_c = \{x \ \in \mathrm{R}^2 | V(x) \leq c\}$ such that $\Omega_c \ \subset \mathrm{D}$

C< min $_{|x_1|=1}$ V(x) =1/4.

As estimate for RoA is $\{ x \ \in \mathrm{R}^2 | V(x) \leq 1/4\}$