Computational Intelligence
Lecture 10: Fuzzy Sets

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Classical Set

Fuzzy Set
- Basic Concepts in Fuzzy Sets
- Operations on Fuzzy Sets
- Fuzzy Complement
- Fuzzy Union
- Fuzzy Intersection
- Averaging Operator
Classical Set

- A classical (crisp) set \( A \) in the universe of discourse \( U \): can be defined by
  - List method: listing all of its members
  - Rule method: specifying the properties that must be satisfied by the members of the set

\[
A = \{ x \in U | x \text{ meets some conditions} \}
\]

- Membership method: introduces a zero-one membership function (also called characteristic function, discrimination function, or indicator function)

\[
\mu_A = \begin{cases} 
1 & \text{if } x \in A \\ 
0 & \text{if } x \notin A 
\end{cases}
\]
Example: cars in Tehran

- The universe of discourse $U$.
- Set $A$ is the cars with 4 cylinders:

$$A = \{x \in U | x \text{ has 4 cylinders}\} \quad \text{OR}$$

$$\mu_A = \begin{cases} 1 & \text{if } x \in U & \text{and } x \text{ has 4 cylinders} \\ 0 & \text{if } x \in U & \text{and } x \text{ does not have 4 cylinders} \end{cases}$$

- Set $D$ is the car made in Iran
- BUT the distinction between an Iranian car and a non-Iranian a car is not crisp:(
  - Most of them are not completely made in Iran
- So what should we do??!!
Fuzzy Set

- Some sets do not have clear boundaries.
- **Fuzzy set:** In a universe of discourse $U$ is characterized by a membership function $\mu_A(x)$ that takes values in the interval $[0, 1]$.
- In classical sets, the membership function of a classical set can only take zero and one.
- In fuzzy set, the membership function is a continuous function with range $[0, 1]$.
- A fuzzy set $A$ in $U$ is represented by:
  - A set of ordered pairs of a generic element $x$ and its membership value: $A = \{(x, \mu_A(x))| x \in U\}$
  - For continuous $U$: $A = \int_U \mu_A(x)/x$.
  - For discrete $U$: $\mu_A(x)$: $A = \sum_U \mu_A(x)/x$
  - $\int$ and $\sum$ do not represent integral and summation.
  - They denote collection of all points $x \in U$ with the associated membership function $\mu_A(x)$.
Example: cars in Tehran (Cont’d)

- $D$: The set "Iranian cars in Iran,"
  - $\mu_D = p(x)$
    - $p(x)$ is the percentage of the parts of car $x$ made in Iran
    - it takes values from 0% to 100%.
- $F$: The set "non-Iranian cars in Iran,"
  - $\mu_F(x) = 1 - p(x)$
Different membership functions can be defined to characterize the same description.
The membership functions are not fuzzy, themselves.
They are precise mathematical functions.
Fuzzy sets are used to defuzzify the world.
**How to determine the membership functions?**
- Formulate human knowledge
  - Usually, gives a rough formula of the membership function
  - fine-tuning is required.
- Data collected from various sensors
  - specify the structures of the membership functions and then fine-tune the parameters based on the data.
A fuzzy set has a one-to-one correspondence with its membership function
Example: Old and Young

- $U$ is in the interval of $[0, 100]$
- $young = \int_0^{25} \frac{1}{x} + \int_{25}^{100} \left(1 + \left(\frac{x-25}{5}\right)^2\right)^{-1/x}$
- $old = \int_{50}^{100} \left(1 + \left(\frac{x-50}{5}\right)^2\right)^{-1/x}$
Example: A Digital Thermometer

- $T$: the set for desirable temperature
- $U \in [18, 33]$

\[
\mu_T = \frac{0}{20} + \frac{.4}{21} + \frac{.5}{22} + \frac{.7}{23} + \frac{.9}{24} + \frac{1}{25} + \frac{.9}{26} + \frac{.7}{27} + \frac{.5}{28} + \frac{.4}{29} + \frac{.2}{30}
\]
Basic Concepts in Fuzzy Sets

- **Support of a fuzzy set** $A$ in the universe of discourse $U$ is a crisp set that contains all the elements of $U$ that have **nonzero membership values** in $A$: \( \text{supp}_A = \{ x \in U | \mu_A > 0 \} \)
  
  - In the digital thermometer example: \( \text{supp}_A = [21, 30] \)
  
  - **empty fuzzy set**: support is empty
  
  - **fuzzy singleton**: support is a single point

- **Center of a fuzzy set**:
  
  - If the **mean value** of all points at which the membership function of the fuzzy set achieves its maximum value is **finite**, then this mean value is the center.
  
  - If the **mean value** equals positive (negative) **infinite**, then the center is the smallest (largest) among all points that achieve the maximum membership value.
- **Crossover point of a fuzzy set**: the point in $U$ whose membership value in $A$ equals 0.5.

- **Height of a fuzzy set**: the largest membership value attained by any point.
  - Normal fuzzy set: the height of fuzzy set equals to one (digital thermometer).

- **$\alpha$-cut of a fuzzy set** $A$ a crisp set $A_\alpha$ contains all the elements in $U$ that have membership values in $A$ greater than or equal to $\alpha$:
  $$A_\alpha = \{ x \in U | \mu_A(x) \geq \alpha \}$$
  - In digital thermometer for $\alpha = 0.7$, $T_\alpha = [23, 24, 25, 26, 27]$
  - A fuzzy set $A$ is **convex** iff its $\alpha$-cut is a convex set for $\forall \alpha \in (0, 1]$. 
Crossover point of a fuzzy set: the point in $U$ whose membership value in $A$ equals 0.5.

Height of a fuzzy set: the largest membership value attained by any point.

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- A fuzzy set $A$ is convex iff its $\alpha$-cut is a convex set for $\forall \alpha \in (0, 1]$.

- In Euclidean space, an object is convex if for every pair of points within the object, every point on the straight line segment that joins them is also within the object.
- Crossover point of a fuzzy set: the point in $U$ whose membership value in $A$ equals 0.5.

- Height of a fuzzy set: the largest membership value attained by any point.
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  - In digital thermometer for $\alpha = 0.7$, $T_\alpha = [23, 24, 25, 26, 27]$
  - A fuzzy set $A$ is convex iff its $\alpha$-cut is a convex set for $\forall \alpha \in (0, 1]$.
    - Let $C$ be a set in a real or complex vector space. $C$ is convex if, $\forall x, y \in C$ and all $\lambda \in [0, 1] \mapsto \lambda x + (1 - \lambda)y \in C$
- **Crossover point of a fuzzy set**: the point in $U$ whose membership value in $A$ equals $0.5$.

- **Height of a fuzzy set**: the largest membership value attained by any point.
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- **$\alpha$-cut of a fuzzy set** $A$ a crisp set $A_\alpha$ contains all the elements in $U$ that have membership values in $A$ greater than or equal to $\alpha$: $A_\alpha = \{x \in U | \mu_A(x) \geq \alpha\}$
  - In digital thermometer for $\alpha = 0.7$, $T_\alpha = [23, 24, 25, 26, 27]$.
  - A fuzzy set $A$ is convex iff its $\alpha$-cut is a convex set for $\forall \alpha \in (0, 1]$.
  - **Lemma**: A fuzzy set $A \in \mathcal{R}^n$ is convex iff
    \[
    \mu_A[\lambda x_1 + (1 - \lambda) x_2] \geq \min[\mu_A(x_1), \mu_A(x_2)] \quad \forall x_1, x_2 \in \mathcal{R}^n, \lambda \in [0, 1].
    \]
Operations on Fuzzy Sets

- **Sets** $F$ and $D$ are equal iff
  \[ \mu_F(x) = \mu_D(x), \forall x \in U \]

- **Set** $D$ contains set $F$ ($F \subset D$), iff
  \[ \mu_F(x) \leq \mu_D(x), \forall x \in U \]

- **Complement of $F$** is a fuzzy set $\bar{F} \in U$
  whose membership function is
  \[ \mu_{\bar{F}}(x) = 1 - \mu_F(x) \]

- **Union** of sets $F$ and $D$ ($F \cup D$) is a fuzzy set in $U$:
  \[ \mu_{F \cup D} = \max[\mu_F(x), \mu_D(x)] \]
  - $F \cup D$ is the smallest fuzzy set containing both $F$ and $D$.

- **Intersection** of $F$ and $D$ ($F \cap D$) is a fuzzy set in $U$:
  \[ \mu_{F \cap D} = \min[\mu_F(x), \mu_D(x)] \]
  - $F \cap D$ is the smallest fuzzy set contained by $F$ and $D$.  

\[ \begin{array}{c}
\text{Graph 1: } \mu_F(x) \quad \mu_{\bar{F}}(x) \\
\text{Graph 2: } \mu_{F \cup D}(x) \\
\text{Graph 3: } \mu_{F \cap D}(x)
\end{array} \]
The De Morgan’s Laws are true for fuzzy sets:

\[ \overline{F \cup D} = \overline{F} \cap \overline{D} \]
\[ \overline{F \cap D} = \overline{F} \cup \overline{D} \]

For Iranian Cars example:

\[ \mu_{F \cup D} = \begin{cases} 
\mu_D & \text{if } 0 \leq p(x) \leq 0.5 \\
\mu_F & \text{if } 0.5 \leq p(x) \leq 1 
\end{cases} \]

\[ \mu_{F \cap D} = \begin{cases} 
\mu_F & \text{if } 0 \leq p(x) \leq 0.5 \\
\mu_D & \text{if } 0.5 \leq p(x) \leq 1 
\end{cases} \]
Further Operations

- An other difference between fuzzy sets and crisp sets:
  - for crisp sets only one type of operation is defined for complement, union, and intersection
  - for fuzzy sets, we can define several operations for them based on the given axioms.
- Why do we need different type of operations?
  - Some operations may not be satisfactory in some situations.
Fuzzy Complement

- Let \( c : [0, 1] \rightarrow [0, 1] \) be a mapping that transforms the membership function of fuzzy set \( A \) into the membership function of the complement of \( A \): \( c[\mu_A(x)] = \mu_{\bar{A}}(x) \)
- It was defined: \( c[\mu_A(x)] = 1 - \mu_A \)
- Let \( a = \mu_A(x_1) \) and \( b = \mu_A(x_2) \)
- the function \( c \) is qualified as a complement if:
  - Axiom c1: \( c(0) = 1 \) and \( c(1) = 0 \) (boundary condition)
  - Axiom c2: \( \forall a, b \in [0, 1], \) if \( a < b \), then \( c(a) \geq c(b) \) (nonincreasing condition)
    - an increase in membership value must result in a decrease or no change in membership value for the complement
Fuzzy Complement

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    - an increase in membership value must result in a decrease or no change in membership value for the complement

- Some types of fuzzy complement:
  - Sugeno class: \( c_\lambda(a) = \frac{1-a}{1+\lambda a}, \quad \lambda \in (-1, \infty) \)
    - \( \lambda = 0 \mapsto \) basic fuzzy complement
  - Yager class: \( c_w(a) = (1 - a^w)^{1/w}, \quad w \in (0, \infty) \)
    - \( w = 1 \mapsto \) basic fuzzy complement
Fuzzy set-S Norm

- Let \( s : [0, 1] \times [0, 1] \rightarrow [0, 1] \) be a mapping that transforms the membership functions of fuzzy sets \( A \) and \( B \) into the membership function of the union of \( A \) and \( B \), that \( s[\mu_A(x), \mu_B(x)] = \mu_{A \cup B} \).

- the function \( S \) to be qualified as an union

- Let \( a = \mu_A(x) \) and \( b = \mu_B(x) \)
  - **Axiom s1.** \( s(1, 1) = 1, s(0, a) = s(a, 0) = a \) (boundary condition).
  - **Axiom s2.** \( s(a, b) = s(b, a) \) (commutative condition).
  - **Axiom s3.** If \( a \leq a' \) and \( b \leq b' \), then \( s(a, b) \leq s(a', b') \) (nondecreasing condition).
  - **Axiom s4.** \( s(s(a, b), c) = s(a, s(b, c)) \) (associative condition).

- Popular types of \( s \)-norm
  - Dombi class: \( s_\lambda(a, b) = \frac{1}{1+[(\frac{1}{a} - 1) - \lambda + (\frac{1}{b} - 1) - \lambda]^{-1/\lambda}}, \lambda \in (0, \infty) \)
  - Dobios-Prade class: \( s_\alpha(a, b) = \frac{a+b-ab-min(a,b,1-\alpha)}{max(1-a,1-b,\alpha)} , \alpha \in [0, 1] \)
  - Yager class: \( s_w(a, b) = min[1, (a^w + b^w)^{1/w}], \ w \in (0, \infty) \)
Other type of s-norm

- Drastic sum: \( s_{ds}(a, b) = \begin{cases} 
  a & \text{if } b = 0 \\
  b & \text{if } a = 0 \\
  1 & \text{otherwise}
\end{cases} \)

- Einstein sum: \( s_{es}(a, b) = \frac{a+b}{1+ab} \)

- Algebric sum: \( s_{as}(a, b) = a + b - ab \)

**Theorem:** *For any s-norm s, that is for any function s : [0, 1] × [0, 1] → [0, 1] that satisfies Axioms s1-s4, the smallest s-norm is maximum and the largest is drastic s-norm*

**Proof:**

- Axioms s1 and s3 \( \Rightarrow s(a, b) \geq s(a, 0) = a \)
- Axiom s2 \( \Rightarrow s(a, b) = s(b, a) \geq s(b, 0) = b \)
- \( \therefore s(a, b) \geq \max(a, b) \)
- If \( b = 0 \), Axiom s1 \( \Rightarrow s(a, b) = s(a, 0) = a \Rightarrow s(a, b) = s_{ds}(a, b) \)
- If \( a = 0 \), Axiom s2 \( \Rightarrow s(a, b) = s_{ds}(a, b) \)
- If \( a \neq 0 \& b \neq 0 \), \( s_{ds}(a, b) = 1 \geq s(a, b) \)
- \( \therefore s(a, b) \leq s_{ds}(a, b), \forall a, b \in [0, 1] \)
Example: The Iranian cars

- Using Algebraic sum:
  \[ \mu_{F \cup D} = p(x) + (1 - p(x)) - p(x)(1 - p(x)) = 1 - p(x) + p(x)^2 \]

- Using Yager s-norm, \( w = 3 \):
  \[ \mu_{F \cup D} = \min[1, (p(x)^3 + (1 - p(x))^3)^{1/3}] \]
Lemma 1: For Dombi class s-norm and Drastic class s-norm it can be defined

\[
\lim_{\lambda \to \infty} s_\lambda(a, b) = \max(a, b)
\]

\[
\lim_{\lambda \to 0} s_\lambda(a, b) = s_{ds}(a, b)
\]

Lemma 2: For Yager class s-norm and Drastic class s-norm it can be defined

\[
\lim_{w \to \infty} s_w(a, b) = \max(a, b)
\]

\[
\lim_{w \to 0} s_w(a, b) = s_{ds}(a, b)
\]
Fuzzy Intersection- T-Norm

Let $t : [0,1] \times [0,1] \to [0,1]$ be a mapping that transforms the membership functions of fuzzy sets $A$ and $B$ into the membership function of the union of $A$ and $B$, that $t[\mu_A(x), \mu_B(x)] = \mu_{A \cap B}$.

The function $T$ to be qualified as an intersection:

- **Axiom t1.** $t(0,0) = 0$, $t(1,a) = t(a,1) = 1$ (boundary condition).
- **Axiom t2.** $t(a,b) = t(b,a)$ (commutative condition).
- **Axiom t3.** If $a \leq a'$ and $b \leq b'$, then $t(a,b) \leq t(a',b')$ (nondecreasing condition).
- **Axiom t4.** $t(t(a,b), c) = t(a, t(b, c))$ (associative condition).

Popular types of $t$-norm

- **Dombi class:** $t_\lambda(a,b) = \frac{1}{1 + [(\frac{1}{a} - 1)^\lambda + (\frac{1}{b} - 1)^\lambda]^{1/\lambda}}$, $\lambda \in (0, \infty)$
- **Dobios-Prade class:** $t_\alpha(a,b) = \frac{ab}{\max(a,b,\alpha)}$, $\alpha \in [0,1]$
- **Yager class:** $t_w(a,b) = 1 - \min[1, ((1-a)^w + (1-b)^w)^{1/w}]$, $w \in (0, \infty)$
Other type of t-norm

- Drastic product: \( t_{ds}(a, b) = \begin{cases} 
  a & \text{if } b = 1 \\
  b & \text{if } a = 1 \\
  0 & \text{otherwise}
\end{cases} \)

- Einstein product: \( t_{ep}(a, b) = \frac{ab}{2-(a+b-ab)} \)

- Algebraic product: \( t_{ap}(a, b) = ab \)

**Theorem:** For any t-norm \( t \), that is for any function \( t : [0, 1] \times [0, 1] \rightarrow [0, 1] \) that satisfies Axioms t1-t4, the largest t-norm is minimum and the smallest is drastic t-norm

prove it.

**Lemma 3:** For Dombi class t-norm and Drastic class t-norm it can be defined

\[
\lim_{\lambda \to \infty} t_\lambda(a, b) = \min(a, b) \\
\lim_{\lambda \to 0} t_\lambda(a, b) = t_{dp}(a, b)
\]
Example: One more time, the Iranian cars

- Using Algebric product:
  \[ \mu_{F \cap D} = p(x)(1 - p(x)) \]

- Using Yager t-norm, \( w = 3 \):
  \[ \mu_{F \cap D} = 1 - \min[1, ((1 - p(x))^3 + p(x)^3)^{1/3}] \]
If the s-norm $s(a, b)$, t-norm $t(a, b)$ and fuzzy complement $c(a)$ satisfy the following equation, they form an associated class (DeMorgan’s Law)

$$c[s(a, b)] = t[c(a), c(b)]$$

**Example:** Show that the Yager s-norm and t-norm with the basic complement are associated

- $c[s_w(a, b)] = 1 - \min[1, (a^w + b^w)^{1/w}]$
- $t_w[c(a), c(b)] = 1 - \min[1, ((1 - 1 + a)^w + (1 - 1 + b)^w)^{1/w}]$
Outline

Classical Set

- minimum
- drastic product
- Einstein product
- algebraic product

- Dombi t-norm
  - $0 \rightarrow \lambda \rightarrow \infty$
- Yager t-norm
  - $0 \rightarrow \omega \rightarrow \infty$

- max(a,b)
- intersection operators

Fuzzy Set

- maximum
- drastic sum
- Einstein sum
- algebraic sum

- Dombi s-norm
  - $\infty \rightarrow \lambda \rightarrow 0$
- Yager s-norm
  - $\infty \rightarrow w \rightarrow 0$

- $s_{ds}(a,b)$
- union operators
Averaging Operator

- This operator fills the gap between \( \min(a, b) \), and \( \max(a, b) \)

- Some average operators:
  - Max-min average: \( v_\lambda(a, b) = \lambda \max(a, b) + (1 - \lambda) \min(a, b) \), \( \lambda \in [0, 1] \)
  - Generalized means: \( v_\alpha(a, b) = \left( \frac{a^\alpha + b^\alpha}{2} \right)^{1/\alpha} \), \( \alpha \in \mathbb{R}, \ \alpha \neq 0 \)
  - Fuzzy and: \( v_p(a, b) = p\min(a, b) + \frac{(1-p)(a+b)}{2} \), \( p = \in [0, 1] \)
  - Fuzzy or: \( v_\gamma(a, b) = \gamma \max(a, b) + \frac{(1-\gamma)(a+b)}{2} \), \( \gamma \in [0, 1] \)
Full Scope of Fuzzy Operators