Signals and Systems
Lecture 9: Z Transform

Farzaneh Abdollahi

Department of Electrical Engineering

Amirkabir University of Technology

Winter 2014

Introduction

Relation Between LT and ZT
- ROC Properties
- The Inverse of ZT
- ZT Properties

Analyzing LTI Systems with ZT

Geometric Evaluation
- LTI Systems Description

Unilateral ZT

- Z transform (ZT) is extension of DTFT
- Like CTFT and DTFT, ZT and LT have similarities and differences.
- We had defined $x[n] = z^n$ as a basic function for DT LTI systems,s.t. $z^n \rightarrow H(z)z^n$
- In Fourier transform $z = e^{j\omega}$, in other words, $|z| = 1$
- In Z transform $z = re^{j\omega}$
- By ZT we can analyze wider range of systems comparing to Fourier Transform
- The bilateral ZT is defined:

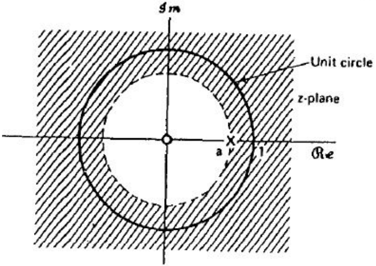
$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ \Rightarrow X(re^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\}e^{-j\omega n} \\ &= \mathcal{F}\{x[n]r^{-n}\} \end{aligned}$$

Region of Convergence (ROC)

- Note that: $X(z)$ exists only for a specific region of z which is called Region of Convergence (ROC)
- ROC: is the $z = re^{j\omega}$ by which $x[n]r^{-n}$ converges:
  $ROC : \{z = re^{j\omega} \ s.t. \ \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty\}$
  - Roc does not depend on ω
  - Roc is absolute summability condition of $x[n]r^{-n}$
- If $r = 1$, i,e, $z = e^{j\omega} \rightsquigarrow X(z) = \mathcal{F}\{x[n]\}$
- ROC is shown in z-plane

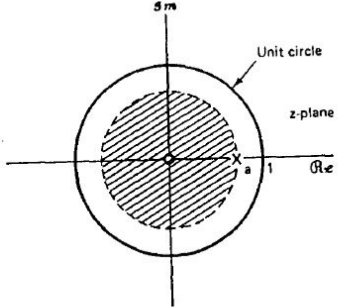
Example

- Consider $x[n] = a^n u[n]$
- $X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$
- If $|z| > |a| \rightsquigarrow X(z)$ is bounded
- $\therefore X(z) = \frac{z}{z-a}, \ ROC : |z| > |a|$



Example

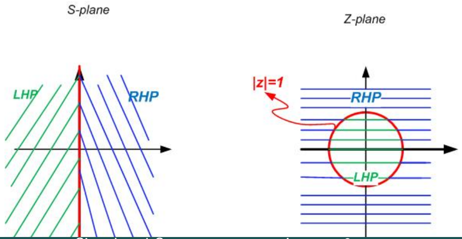
- Consider $x[n] = -a^n u[-n-1]$
- $X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^{-n}$
- If $|a^{-1}z| < 1 \rightsquigarrow |z| < |a|, X(z)$ is bounded
- $\therefore X(z) = \frac{z}{z-a}, \; ROC: |z| < |a|$



- In the recent two examples two different signals had similar ZT but with different ROC
- To obtain unique $x[n]$ both $X(z)$ and ROC are required
- If $X(z) = \frac{N(z)}{D(z)}$
  - Roots of $N(z)$ zeros of X(z); They make X(z) equal to zero.
  - Roots of $D(z)$ poles of X(z); They make X(z) to be unbounded.

Relation Between LT and ZT

- In LT: $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = \int_{n=-\infty}^{\infty} x(t)e^{-st}dt = \mathcal{L}\{x(t)\}$
- Now define $t = nT$:
  $X(s) = \lim_{T\to 0} \sum_{n=-\infty}^{\infty} x(nT)(e^{sT})^{-n}.T =$
  $\lim_{T\to 0} T \sum_{n=-\infty}^{\infty} x[n](e^{sT})^{-n}$
- In ZT: $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathcal{Z}\{x[n]\}$
- $\therefore$ by taking $z = e^{sT}$ ZT is obtained from LT.
- $j\omega$ axis in s-plane is changed to unite circle in z-plane



z-Plane	s-Plane
$\|z\| < 1$ (insider the unit circle) special case: $\|z\| = 0$	$\mathcal{R}e\{s\} < 0$ (LHP) special case: $\mathcal{R}e\{s\} = -\infty$
$\|z\| > 1$ (outsider the unit circle) special case: $\|z\| = \infty$	$\mathcal{R}e\{s\} > 0$ (RHP) special case: $\mathcal{R}e\{s\} = \infty$
$\|z\| = cte$ (a circle)	$\mathcal{R}e\{s\} = cte$ (a vertical line)

ROC Properties

- ROC of $X(z)$ is a ring in z-plane centered at origin
- ROC does not contain any pole
- If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$
  - $X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n}$
  - If $N_1 < 0 \rightsquigarrow x[n]$ has nonzero terms for $n < 0$, when $|z| \to \infty$ positive power of z will be unbounded
  - If $N_2 > 0 \rightsquigarrow x[n]$ has nonzero terms for $n > 0$, when $|z| \to 0$ negative power of z will be unbounded
  - If $N_1 \geq 0 \rightsquigarrow$ only negative powers of z exist $\rightsquigarrow z = \infty \in$ ROC
  - If $N_2 \leq 0 \rightsquigarrow$ only positive powers of z exist $\rightsquigarrow z = 0 \in$ ROC
  - **Example:**

ZT	LT
$\delta[n] \leftrightarrow 1$ ROC: all z	$\delta(t) \leftrightarrow 1$ ROC: all s
$\delta[n-1] \leftrightarrow z^{-1}$ ROC: $z \neq 0$	$\delta(t-T) \leftrightarrow e^{-sT}$ ROC: $\mathcal{R}e\{s\} \neq -\infty$
$\delta[n+1] \leftrightarrow z$ ROC: $z \neq \infty$	$\delta(t+T) \leftrightarrow e^{sT}$ ROC: $\mathcal{R}e\{s\} \neq \infty$

ROC Properties

- If $x[n]$ is a right-sided sequence and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC.
- If $x[n]$ is a left-sided sequence and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| < r_0$ will also be in the ROC.
- If $x[n]$ is a two-sided sequence and if the circle $|z| = r_0$ is in the ROC, then ROC is a ring containing $|z| = r_0$.
- If $X(z)$ is rational
  - The ROC is bounded between poles or extends to infinity,
  - no poles of $X(s)$ are contained in ROC
  - If $x[n]$ is right sided, then ROC is in the out of the outermost pole
    - If $x[n]$ is causal and right sided then $z = \infty \in$ ROC
  - If $x[n]$ is left sided, then ROC is in the inside of the innermost pole
    - If $x[n]$ is anticausal and left sided then $z = 0 \in$ ROC
- If ROC includes $|z| = 1$ axis then $x[n]$ has FT

The Inverse of Z Transform (ZT)

- By considering r fixed, inverse of ZT can be obtained from inverse of FT:
- $x[n]r^{-n} = \frac{1}{2\pi}\int_{2\pi} X(\underbrace{re^{j\omega}}_{z})e^{j\omega n}d\omega$
- $x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})r^n e^{(j\omega)n}d\omega$
- assuming r is fixed $\rightsquigarrow dz = jzd\omega$
- $\therefore x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$
- Methods to obtain Inverse ZT:
  1. If $X(z)$ is rational , we can use expanding the rational algebraic into a linear combination of lower order terms and then one may use
     - $X(z) = \frac{A_i}{1-a_i z^{-1}} \rightsquigarrow x[n] = A_i a_i^n u[n]$ if ROC is out of pole $z = a_i$
     - $X(z) = \frac{A_i}{1-a_i z^{-1}} \rightsquigarrow x[n] = -A_i a_i^n u[-n-1]$ if ROC is inside of $z = a_i$

  Do not forget to consider ROC in obtaining inverse of ZT!

Methods to obtain Inverse ZT:

2. If X is nonrational, use Power series expansion of X(z), then apply $\delta[n+n_0] \Leftrightarrow z^{n_0}$
   - **Example:** $X(z) = 5z^2 - z + 3z^{-3}$
   - $x[n] = 5\delta[n+2] - \delta[n+1] + 3\delta[n-3]$

3. If X is rational, power series can be obtained by long division
   - **Example:** $X(z) = \frac{1}{1-az^{-1}}, \; |z| > |a|$

$$\begin{array}{r|l} 1 & 1-az^{-1} \\ \hline & 1+az^{-1}+(az^{-1})^2+\ldots \\ \underline{-1+az^{-1}} & \\ az^{-1} & \\ \underline{-az^{-1}+a^2z^{-2}} & \\ \vdots & \end{array}$$

   - $x[n] = a^n u[n]$

Methods to obtain Inverse ZT:

- **Example:** $X(z) = \frac{1}{1-az^{-1}}, \ |z| < |a|$
- $X(z) = -a^{-1}z(\frac{1}{1-a^{-1}z})$

$$
\begin{array}{r|l}
1 & 1-a^{-1}z \\ \cline{2-2}
\underline{-1+a^{-1}z} & 1+a^{-1}z+(a^{-1}z)^2+\ldots \\
a^{-1}z & \\
\underline{-a^{-1}z+a^{-2}z^2} & \\
\vdots &
\end{array}
$$

- $x[n] = -a^n u[-n-1]$

ZT Properties

- Linearity: $ax_1[n] + bx_2[n] \Leftrightarrow aX_1(z) + bX_2(z)$
  - ROC contains: $R_1 \bigcap R_2$
  - If $R_1 \bigcap R_2 = \emptyset$ it means that ZT does not exit
  - By zeros and poles cancelation ROC can be lager than $R_1 \bigcap R_2$
- Time Shifting:$x[n - n_0] \Leftrightarrow z^{-n_0} X(z)$ with ROC$=R$ (maybe 0 or ∞ is added/omited)
  - If $n_0 > 0 \rightsquigarrow z^{-n_0}$ may provide poles at origin $\rightsquigarrow 0 \in$ ROC
  - If $n_0 < 0 \rightsquigarrow z^{-n_0}$ may eliminate ∞ from ROC
- Time Reversal: $x[-n] \Leftrightarrow X(\frac{1}{z})$ with ROC$= \frac{1}{R}$
- Scaling in Z domain: $z_0^n x[n] \Leftrightarrow X(\frac{z}{z_0})$ with ROC $= |z_0|R$
  - If $X(z)$ has zero/poles at $z = a \rightsquigarrow X(\frac{z}{z_0})$ has zeros/poles at $z = z_0 a$
- Differentiation in the z-Domain: $nx[n] \Leftrightarrow -z\frac{dX(z)}{dz}$ with ROC $= R$
- Convolution: $x_1[n] * x_2[n] \Leftrightarrow X_1(z)X_2(z)$ with ROC containing $R_1 \cap R_2$

ZT Properties

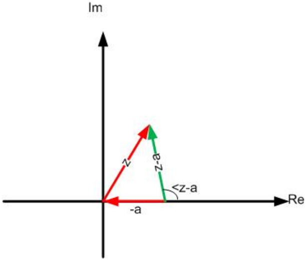
- Conjugation: $x^*[n] \Leftrightarrow X^*(z^*)$ with ROC $= R$
  - If $x[n]$ is real
    - $X(z) = X^*(z^*)$
    - If $X(z)$ has zeros/poles at z_0 it should have zeros/poles at z_0^* as well
- Initial Value Theorem: If $x[n] = 0$ for $n < 0$ and $x[0]$ is bounded, then $x[0] = \lim_{z\to\infty} X(z)$
  - $\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + \lim_{z\to\infty} \sum_{n=1}^{\infty} x[n]z^{-n} = x[0]$
  - For a casual $x[n]$, if $x[0]$ is bounded it means # of zeros are less than or equal to # of poles (This is true for CT as well)
- Final Value Theorem: If $x[n] = 0$ for $n < 0$ and $x[n]$ is bounded when $n \to \infty$ then $x[\infty] = \lim_{z\to 1}(1 - z^{-1})X(z)$

Analyzing LTI Systems with ZT

- ZT of impulse response is $H(z)$ which is named transfer function or system function.
- Transfer fcn can represent many properties of the system:
  - Casuality: $h[n] = 0$ for $n < 0 \rightsquigarrow$ It is right sided
    - ROC of a causal LTI system is out of a circle in z-plane, it includes ∞
    - Note that the converse is not always correct
    - For a system with rational transfer fcn, causality is equivalent to ROC being outside of the outermost pole (degree of nominator should not be greater than degree of denominator)
  - Stability: $h[n]$ should be absolute summable $\rightsquigarrow$ its FT converges
    - An LTI system is stable iff its ROC includes unit circle
  - A causal system with rational $H(z)$ is stable iff all the poles of $H(z)$ are inside the unit circle
- DC gain in DT is $H(1)$ and in CT is $H(0)$

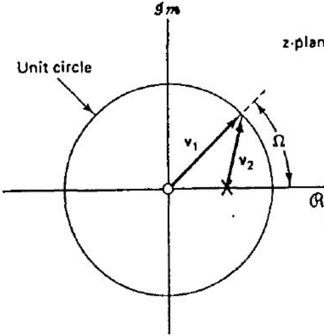
Geometric Evaluation of FT by Zero/Poles Plot

- Consider $X_1(z) = z - a$



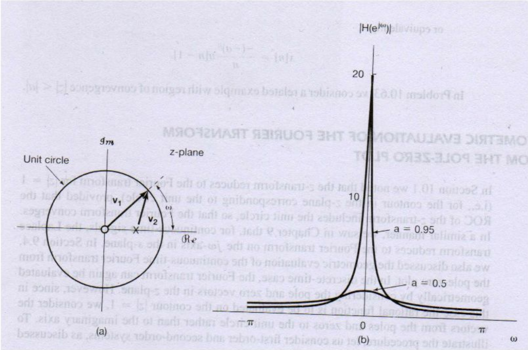
  - $|X_1|$: length of X_1
  - $\measuredangle X_1$: angel of X_1
- Now consider $X_2(z) = \frac{1}{z-a} = \frac{1}{X_1(z)}$
  - $|X_2| = \frac{1}{|X_1(z)|}$
  - $\measuredangle X_2 = -\measuredangle X_1$

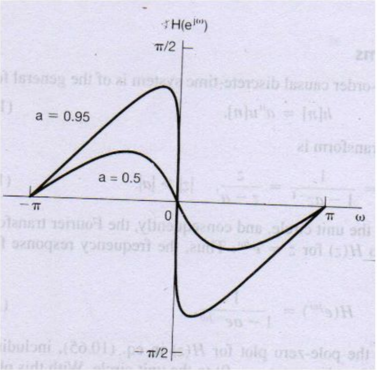
- For higher order fcns:
  $X(z) = M\frac{\prod_{i=1}^{R}(z-\beta_i)}{\prod_{j=1}^{P}(z-\alpha_j)}$
  - $|X(z)| = |M|\frac{\prod_{i=1}^{R}|z-\beta_i|}{\prod_{j=1}^{P}|z-\alpha_j|}$
  - $\measuredangle X(z) = \measuredangle M + \sum_{i=1}^{R}\measuredangle(z-\beta_i) - \sum_{j=1}^{R}\measuredangle(z-\alpha_j)$
- Example: $H(z) = \frac{1}{1-az^{-1}}$, $|z| > |a|, |a| < 1$, a real
  - $h(t) = a^n u[n]$
  - $H(e^{j\omega}) = \frac{v_1}{v_2}$, $|H(e^{j\omega})| = \frac{1}{|v_2|}$
    - at $\omega = 0, |H(e^{j\omega})| = \frac{1}{1-a}$ is max
    - $0 < \omega < \pi$: $\omega \uparrow \rightsquigarrow |H(e^{j\omega})| \downarrow$
    - at $\omega = \pi, |H(e^{j\omega})| = \frac{1}{1+a}$
  - $\measuredangle H(e^{j\omega}) = \angle v_1 - \angle v_2 = \omega - \angle v_2$
    - at $\omega = 0, \angle H(e^{j\omega}) = 0$
    - $0 < \omega < \pi, \omega \uparrow \rightsquigarrow \angle H(e^{j\omega}) \downarrow$
    - at $\omega = \pi, \angle H(e^{j\omega}) = 0$



First Order Systems

- a in DT first order systems plays a similar role of time constant τ in CT
- $|a|\downarrow$
  - $|H(e^{j\omega})|\downarrow$ at $\omega = 0$
  - Impulse response decays more rapidly
  - Step response settles more quickly
- In case of having multiple poles, the poles closer to the origin, decay more rapidly in the impulse response

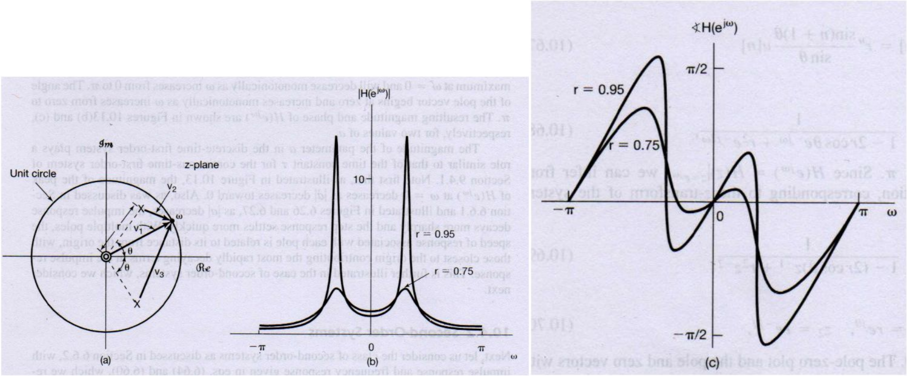

|H(e^{jω})|
20
10
a = 0.95
a = 0.5
Unit circle
z-plane
v1
v2
ω
−π
0
π
(a)
(b)
ω


∢H(e^{jω})
π/2
a = 0.95
a = 0.5
−π
0
π
ω
−π/2

Second Order Systems

- Consider a second order system with poles at $z_1 = re^{j\theta}, z_2 = re^{-j\theta}$
- $H(z) = \frac{z^2}{(z-z_1)(z-z_2)} = \frac{1}{1-2rcos\theta z^{-1}+r^2z^{-2}}, 0 < r < 1, 0 < \theta < \pi$
- $h[n] = \frac{r^n sin(n+1)\theta}{sin\theta} u[n]$
- $\angle H = 2\angle v_1 - \angle v_2 - \angle v_3$
- Starting from $\omega = 0$ to $\omega = \pi$:
  - At first v_2 is decreasing
  - v_2 is min at $\omega = \theta \rightsquigarrow$ max $|H|$
  - Then v_2 is increasing
- If r is closer to unit, then $|H|$ is greater at poles and change of $\angle H$ is sharper
- If r is closer to the origin, impulse repose decays more rapidly and step response settles more quickly.

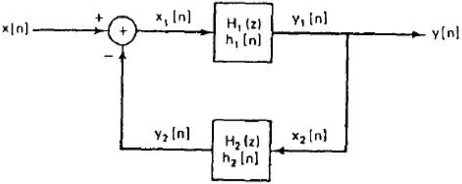
Bode Plot of $H(j\omega)$



LTI Systems Description

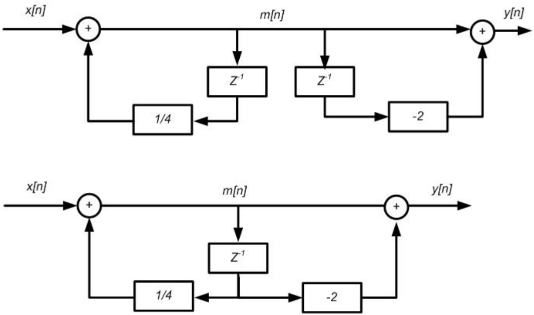
- $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$
- $\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$
- $H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$
- ROC depends on
  - placement of poles
  - boundary conditions (right sided, left sided, two sided,...)

- Feedback Interconnection of two LTI systems:
  - $Y(z) = Y_1(z) = X_2(z)$
  - $X_1(z) = X(z) - Y_2(z) = X(z) - H_2(z)Y(z)$
  - $Y(z) = H_1(z)X_1(z) = H_1(z)[X(z) - H_2(z)Y(z)]$
  - $\frac{Y(z)}{X(z)} = H(z) = \frac{H_1(z)}{1+H_2(z)H_1(z)}$
  - ROC: is determined based on roots of $1 + H_2(z)H_1(z)$



Block Diagram Representation for Casual LTI Systems

- We can represent a transfer fcn by different methods:
- Example: $H(z) = \frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}} = (\frac{1}{1-\frac{1}{4}z^{-1}})(1-2z^{-1})$



Unilateral ZT

- It is used to describe casual systems with nonzero initial conditions: $\mathcal{X}(z) = \sum_{0^-}^{\infty} x[n]z^{-n} = \mathcal{UZ}\{x[n]\}$
- If $x[n] = 0$ for $n < 0$ then $\mathcal{X}(z) = X(z)$
- Unilateral ZT of $x[n] =$ Bilateral ZT of $x[n]u[n]$
- If $h[n]$ is impulse response of a casual LTI system then $H(z) = \mathcal{H}(z)$
- ROC is not necessary to be recognized for unilateral ZT since it is always outside of a circle
- For rational $\mathcal{X}(z)$, ROC is outside of the outmost pole

Similar Properties of Unilateral and Bilateral ZT

- Convolution: Note that for unilateral ZT, If both $x_1[n]$ and $x_2[n]$ are zero for $t < 0$, then $\mathcal{X}(z) = \mathcal{X}_1(z)\mathcal{X}_2(z)$
- Time Scaling
- Time Expansion
- Initial and Finite Theorems: they are indeed defined for causal signals
- Differentiating in z domain:
- The main difference between $\mathcal{UL}$ and ZT is in time differentiation:
  - $\mathcal{UZ}\{x[n-1]\} = \sum_0^\infty x[n-1]z^{-n} = x[-1] + \sum_{n=1}^\infty x[n-1]z^{-n} = x[-1] + \sum_{m=0}^\infty x[m]z^{-m-1}$
  - $\mathcal{UZ}\{x[n-1]\} = x[-1] + z^{-1}\mathcal{X}(z)$
  - $\mathcal{UZ}\{x[n-2]\} = x[-2] + z^{-1}x[-1] + z^{-2}\mathcal{X}(z)$
  - $\mathcal{UZ}\{x[n+1]\} = \sum_{n=0}^\infty x[n+1]z^{-n} = \sum_{m=1}^\infty x[m]z^{-m+1} \pm x[0]z$
  - $\mathcal{UZ}\{x[n+1]\} = z\mathcal{X}(z) - zx[0]$
  - $\mathcal{UZ}\{x[n+2]\} = z^2\mathcal{X}(z) - z^2x[0] - zx[1]$
  - Follow the same rule for higher orders

Example

- Consider $y[n] + 2y[n-1] = x[n]$, where $y[-1] = \beta,\ x[n] = \alpha u[n]$
- Take $\mathcal{UZ}$:
  - $\mathcal{Y}(z) + 2[\beta + z^{-1}\mathcal{Y}(z)] = \mathcal{X}(z)$
  - $\mathcal{Y}(z) = \underbrace{\frac{-2\beta}{1+2z^{-1}}}_{ZIR} + \underbrace{\frac{\mathcal{X}(z)}{1+2z^{-1}}}_{ZSR}$
  - Zero State Response (ZSR): is a response in absence of initial values
    - $\mathcal{H}(z) = \frac{\mathcal{Y}(z)}{\mathcal{X}(z)}$
    - Transfer fcn is ZSR
    - ZSR: $\mathcal{Y}_1(z) = \frac{\alpha}{(1+2z^{-1})(1-z^{-1})}$
  - Zero Input Response (ZIR): is a response in absence of input ($x[n] = 0$)
  - ZIR: $\mathcal{Y}_2(z) = \frac{-2\beta}{1+2z^{-1}}$
  - $y_2[n] = -2\beta(-2)^n u[n]$
- $y[n] = y_1[n] + y_2[n]$