Nonlinear Control
Lecture 10: Back Stepping

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More General Form

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Integrator Back Stepping

- Let us start with integrator back stepping:

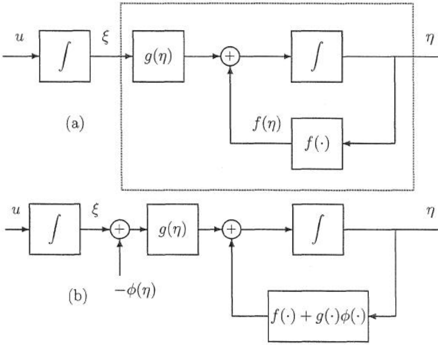
$$\dot{\eta} = f(\eta) + g(\eta)\varepsilon \quad (1)$$

$$\dot{\varepsilon} = u \quad (2)$$

  - $[\eta^T \ \varepsilon]^T \in R^{n+1}$: is the state
  - $u \in R$: control input
  - $f : D \rightarrow R^n$ and $g : D \rightarrow R^n$: smooth in a domain $D \subset R^n$; $\eta = 0, f(0) = 0$
- Objective: Design a state FB controller to stabilize the origin $(\eta = 0, \varepsilon = 0)$
- We assume both f and g are known
- It is a cascade connection:
  - (1) with input ε
  - Second is the integrator (2)

- Suppose (1) can be asym. stabilized by $\varepsilon = \phi(\eta)$ with $\phi(0) = 0$:
  $\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$
- and $V(\eta)$ is a smooth p.d. Lyap fcn:
  $\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta) \quad \forall \eta \in D$, $W(\eta)$ is p.d.
- Now add $\pm g(\eta)\phi(\eta)$ to (1):

$$
\begin{aligned}
\dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] \\
&+ g(\eta)[\varepsilon - \phi(\eta)] \\
\dot{\varepsilon} &= u
\end{aligned}
$$



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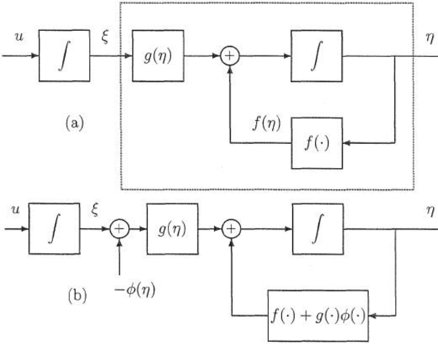
- Now add $\pm g(\eta)\phi(\eta)$ to (1):

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] \\ &+ g(\eta)[\underbrace{\varepsilon - \phi(\eta)}_{z}] \end{aligned}$$



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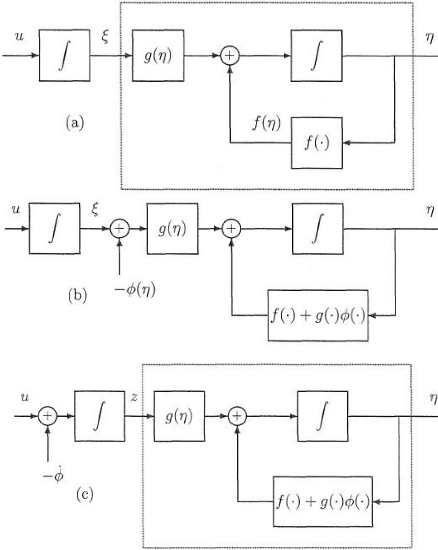
- Now add $\pm g(\eta)\phi(\eta)$ to (1):

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] \\ &+ g(\eta)\underbrace{[\varepsilon - \phi(\eta)]}_{z} \end{aligned}$$

$$\begin{aligned} \therefore \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] \\ &+ g(\eta)z \\ \dot{z} &= u - \dot{\phi} \\ \dot{\phi} &= \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\varepsilon] \end{aligned}$$



- Fig b to Fig c is back stepping $-\phi$ though the integrator
- $v = u - \dot{\phi} \rightsquigarrow$

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\ \dot{z} &= v \end{aligned}$$

- It is similar to (1) But input zero $\rightsquigarrow$ origin is a.s.
- Now let us design v to stabilize the over all system: $V_c(\eta, \varepsilon) = V(\eta) + \frac{1}{2}z^2$
- $\therefore \dot{V}_c \leq -W(\eta) + \frac{\partial V}{\partial \eta} g(\eta) z + zv$
- Choose $v = -\frac{\partial V}{\partial \eta} g(\eta) - kz, \quad k > 0$
- So $\dot{V}_c \leq -W(\eta) - kz^2$
- $\therefore$ origin is a.s. $(\eta = 0, z = 0)$
- $\phi(0) = 0 \rightsquigarrow \varepsilon = 0$

$$u = \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\varepsilon] - \frac{\partial V}{\partial \eta} g(\eta) - k(\varepsilon - \phi(\eta)) \tag{3}$$

- **Lemma:** *Consider the system (1)-(2). Let $\phi(\eta)$ be a stabilizing state fb control law for (1) with $\phi(0) = 0$, and $V(\eta)$ be a Lyap fcn that $\dot{V} \leq -W(\eta)$ for some p.d fcn $W(\eta)$. Then, the state feedback control law (3) stabilizes the origin of (1)-(2), with $V(\eta) + [\varepsilon - \phi(\eta)]^2/2$ as a Lyap fcn. Moreover, if all the assumptions hold globally and $V(\eta)$ is "radially unbounded", the origin will be g.a.s.*
- **Example:** Consider

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= u \end{aligned}$$

- Therefore $\eta = x_1, \varepsilon = x_2$
- To stabilize $x_1 = 0$: $x_2 = \phi(x_1) = -x_1^2 - x_1$
- $\therefore$ the nonlinear term x_1^2 is canceled: $\dot{x}_1 = -x_1 - x_1^3$
  - Why $-x_1^3$ is not canceled?
- $V(x_1) = x_1^2/2 \rightsquigarrow \dot{V} = -x_1^2 - x_1^4 \leq -x_1^2, \ \forall x_1 \in R$
- $\therefore$ The origin of $\dot{x}_1$ is g.e.s.

- To backstep: $z_2 = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2$
- Hence

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_1^3 + z_2 \\ \dot{z}_2 &= u + (1 + 2x_1)(-x_1 - x_1^3 + z_2) \end{aligned}$$

- Now take $V_c = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$
- $\dot{V}_c = -x_1^2 - x_1^4 + z_2[x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + u]$
- $\therefore u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2 \rightsquigarrow \dot{V}_c = -x_1^2 - x_1^4 - z_2^2$
- The origin is g.a.s

- For higher order systems we can apply the recursive application of integrator back stepping
- **Example:** Consider

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u\end{aligned}$$

- After 1 back stepping:

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ \dot{x}_2 &= x_3\end{aligned}$$

- that x_3 is input is g.s. by:
  $x_3 = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - (x_2 + x_1 + x_1^2) = \phi(x_1, x_2)$
- and $V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2)^2$

- Backstep again: $z_3 = x_3 - \phi(x_1, x_2)$

$$\begin{aligned}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\
\dot{x}_2 &= \phi(x_1, x_2) + z_3 \\
\dot{z}_3 &= u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(\phi + z_3)
\end{aligned}$$

- Define $V_c = V + z_3^2/2 \rightsquigarrow \dot{V}_c =$ $-x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 + z_3[\frac{\partial V}{\partial x_2} - \frac{\partial \phi}{\partial x_1}(x_1^2 + x_x^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi) + u]$
- $\therefore u = -\frac{\partial V}{\partial x_2} + \frac{\partial \phi}{\partial x_1}(x_1^2 + x_x^3 + x_2) + \frac{\partial \phi}{\partial x_2}(z_3 + \phi) - z_3$
- The origin is g.a.s

Back Stepping for More General Form

- Consider

$$\begin{aligned} \dot{\eta} &= f(\eta) + g(\eta)\varepsilon \\ \dot{\varepsilon} &= f_a(\eta, \varepsilon) + g_a(\eta, \varepsilon)u \end{aligned} \tag{4}$$

  - f_a and g_a are smooth
  - If $g_a(\eta, \varepsilon) \neq 0$ over the domain of interest: define

$$u = \frac{1}{g_a(\eta, \varepsilon)}[u_a - f_a(\eta, \varepsilon)]$$

- if a stabilizing state feedback control law $\phi(\eta)$ and a Lyap fcn. $V(\eta)$ exists s.t. satisfy the conditions of Lemma:
  $u = \frac{1}{g_a(\eta,\varepsilon)}[\frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\varepsilon] - \frac{\partial V}{\partial \eta} g(\eta) - k[\varepsilon - \phi(\eta)] - f_a(\eta, \varepsilon)] \quad k > 0$
- and $V_c(\eta, \varepsilon) = V(\eta) + \frac{1}{2}[\varepsilon - \phi(\eta)]^2$

Back Stepping for Strict-Feedback Systems

- By recursive backstepping strict-FB systems can be stabilized:

$$
\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(x, z_1, \ldots, z_{k-1})z_k \\
\dot{z}_k &= f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k)u
\end{aligned}
$$

  - $x \in R^n$
  - z_1 to z_k are scalar
  - $f_0(0)$ to $f_k(0)$ are zero
  - $g_i(x, z_1, ..., z_i) \neq 0$ for $1 \leq i \leq k$ over the domain of interest
- "strict FB" $\equiv$ f_i and g_i in $\dot{z}_i$ **only** depends on $x, z_1, ..., z_i$

- Start the recursive procedure with $\dot{x} = f_0(x) + g_0(x)z_1$
- Determine a stabilizing state fb $z_1 = \phi_0(x), \quad \phi_0(0) = 0$ and $\frac{\partial V_0}{\partial x}[f_0(x) + g_0(x)\phi_0(x)] \leq -W(x), \quad W(x)$ is p.d.
- Apply backstepping, consider

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x)z_1 \\ \dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \end{aligned}$$

- The parameters can be defined as
  $\eta = x, \ \varepsilon = z_1, \ u = z_2, \ f = f_0, \ g = g_0, \ f_a = f_1, \ g_a = g_1$
- The stabilizing state fb:
  $\phi_1(x, z_1) = \frac{1}{g_1}[\frac{\partial \phi_0}{\partial x}(f_0 + g_0 z_1) - \frac{\partial V_0}{\partial x} g_0 - k_1(z_1 - \phi) - f_1], \quad k_1 > 0$
- The Lyap fcn: $V_1(x, z_1) = V_0(x) + \frac{1}{2}[z_1 - \phi_1(x)]^2$

- Now consider:

$$\begin{aligned}\dot{x} &= f_0(x) + g_0(x)z_1 \\ \dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\ \dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3\end{aligned}$$

- The parameters can be defined as $\eta = [x \quad z_1]^T, \ \varepsilon = z_2, \ u = z_3, \ f = [f_0 + g_0 z_1 \quad f_1]^T, \ g = [0 \quad g_1]^T, \ f_a = f_2, \ g_a = g_2$
- The stabilizing state fb: $\phi_2(x, z_1, z_2) = \frac{1}{g_2}[\frac{\partial \phi_1}{\partial x}(f_0 + g_0 z_1) + \frac{\partial \phi_1}{\partial z_1}(f_1 + g_1 z_2) - \frac{\partial V_1}{\partial z_1} g_1 - k_2(z_2 - \phi) - f_2], \quad k_2 > 0$
- The Lyap fcn: $V_2(x, z_1, z_2) = V_1(, z_1 x) + \frac{1}{2}[z_2 - \phi_2(x, z_1)]^2$
- This process should be repeated k times to obtain $u = \phi_k(x, z_1, ..., z_k)$ and Layp fcn $V_k(x, z_1, ..., z_k)$
- **If a system has not defined in strict FB system, one can transform the states by normal transformation**

Back Stepping for Uncertain Systems

- Consider the system:

$$\begin{aligned} \dot{\eta} &= f(\eta) + g(\eta)\varepsilon + \delta_\eta(\eta, \varepsilon) \\ \dot{\varepsilon} &= f_a(\eta, \varepsilon) + g_a(\eta, \varepsilon)u + \delta_\varepsilon(\eta, \varepsilon) \end{aligned} \tag{5}$$

  - in domain $D \subset R^{n+1}$
  - contains $(\eta = 0, \varepsilon = 0)$; $\eta \in R^n, \varepsilon \in R$
  - all fcns are smooth
  - If $g_a(\eta, \varepsilon) \neq 0$ over the domain of interest
  - f, g, f_a, g_a are known; $\delta_\eta, \delta_\varepsilon$ are uncertain terms
  - $f(0) = 0$ and $f_a(0, 0) = 0$

$$\begin{aligned} \|\delta_\eta(\eta, \varepsilon)\|_2 &\leq a_1 \|\eta\|_2 \\ |\delta_\varepsilon(\eta, \varepsilon)| &\leq a_2 \|\eta\|_2 + a_3 |\varepsilon| \end{aligned} \tag{6}$$

  - Note: The upper bound on $\delta_\eta(\eta, \varepsilon)$ only depends on η.

- $V(\eta)$ for the first equation is $\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \varepsilon)] \leq -b\|\eta\|_2^2$; b is pos. const.
- $\therefore \eta = 0$ is a.s. Equ. point of $\dot{\eta} = f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \varepsilon)$
- Suppose $$|\phi(\eta)| \leq a_4\|\eta\|_2, \quad \|\frac{\partial \phi}{\partial \eta}\|_2 \leq a_5 \tag{7}$$
- Consider the Layp fcn for whole system: $V_c(\eta, \varepsilon) = V(\eta) + \frac{1}{2}[\varepsilon - \phi(\eta)]^2$
- $\dot{V}_c = \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \varepsilon)] + \frac{\partial V}{\partial \eta} g(\varepsilon - \phi) + (\varepsilon - \phi)[f_a + g_a u + \delta_\varepsilon - \frac{\partial \phi}{\partial \eta}(f + g\varepsilon + \delta_\eta)]$

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- Choose $u = \frac{1}{g_a}[-f_a + \frac{\partial \phi}{\partial \eta}(f + g\varepsilon) - \frac{\partial V}{\partial \eta} g - k(\varepsilon - \phi)], \quad k > 0$
- $\dot{V}_c \leq -b\|\eta\|_2^2 + 2a_6\|\eta\|_2|\varepsilon - \phi| - (k - a_3)(\varepsilon - \phi)^2 = -\begin{bmatrix} \|\eta\|_2 \\ |\varepsilon - \phi| \end{bmatrix}^T \underbrace{\begin{bmatrix} b & -a_6 \\ -a_6 & (k - a_3) \end{bmatrix}}_{P} \begin{bmatrix} \|\eta\|_2 \\ |\varepsilon - \phi| \end{bmatrix}; \quad a_6 = \frac{a_3 a_4 + a_2 + a_5 a_1}{2}$

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- $\dot{V}_c = \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \varepsilon)] + \frac{\partial V}{\partial \eta} g(\varepsilon - \phi) + (\varepsilon - \phi)[f_a + g_a u + \delta_\varepsilon - \frac{\partial \phi}{\partial \eta}(f + g\varepsilon + \delta_\eta)]$
- Choose $u = \frac{1}{g_a}[-f_a + \frac{\partial \phi}{\partial \eta}(f + g\varepsilon) - \frac{\partial V}{\partial \eta} g - k(\varepsilon - \phi)], \quad k > 0$
- $\dot{V}_c \leq -b\|\eta\|_2^2 + 2a_6\|\eta\|_2|\varepsilon - \phi| - (k - a_3)(\varepsilon - \phi)^2 =$ $$-\begin{bmatrix} \|\eta\|_2 \\ |\varepsilon - \phi| \end{bmatrix}^T \underbrace{\begin{bmatrix} b & -a_6 \\ -a_6 & (k - a_3) \end{bmatrix}}_{P} \begin{bmatrix} \|\eta\|_2 \\ |\varepsilon - \phi| \end{bmatrix}; \quad a_6 = \frac{a_3 a_4 + a_2 + a_5 a_1}{2}$$
- Choose $k > a_3 + \frac{a_6^2}{b} \rightsquigarrow \dot{V}_c \leq -\lambda_{min}(P)[\|\eta\|_2^2 + |\varepsilon - \phi|^2]$

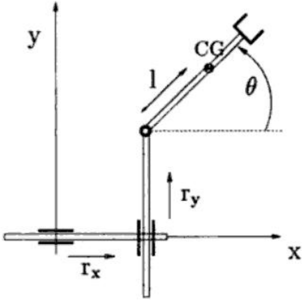
- **Lemma:** *Consider the system (5), where the uncertainty satisfies inequalities (6). Let $\phi(\eta)$ be a stabilizing state fb control law that satisfies (7), and $V(\eta)$ be a Lyap. fcn that guarantee a.s. of the first Equatin of (5). Then the given state feedback control law in previous slide, with k sufficiently large, stabilizes the origin of (5). Moreover, if all the assumptions hold globally and $V(\eta)$ is radially unbounded, the origin will be g.a.s.*

Trajectory Tracking for A Second Order System [1]

- A 3 link underactuated manipulator
  - The first two translational joints are actuated
  - The third revolute joint is not actuated
  - The linear approximation of this system is not controllable since it is not influenced by gravity
- The dynamics:

$$
\begin{aligned}
m_x\ddot{r}_x - m_3 l sin(\theta)\dot{\theta} - m_3 l cos(\theta)\dot{\theta}^2 &= \tau_1 \\
m_y\ddot{r}_y + m_3 l cos(\theta)\dot{\theta} - m_3 l sin(\theta)\dot{\theta}^2 &= \tau_2 \\
I\ddot{\theta} - m_3 l sin(\theta)\ddot{r}_x + m_3 l cos(\theta)\ddot{r})y &= 0 \\
\lambda\ddot{\theta} + \ddot{r}_x sin(\theta) + \ddot{r}_y cos(\theta) &= 0
\end{aligned}
$$

where $[r_x, r_y]$: displacement of third joint; θ orientation of third link respect to x axis; τ_1 τ_2: input of actuated joints; m_i: mass, I_i: inertia; $\lambda = (I_3 + m_3 l^2)/(m_3 l)$



- Transform the dynamics by:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} r_x + \lambda(cos(\theta) - 1) \\ tan(\theta) \\ r_y + \lambda sin(\theta) \end{bmatrix} ; \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} =$$

$$\begin{bmatrix} -m_3 lcos(\theta)\dot{\theta}^2 + (m_x - \frac{I}{\lambda^2} sin^2(\theta))v_x + (\frac{I}{\lambda^2} sin(\theta)cos(\theta))v_y \\ -m_3 lsin(\theta)\dot{\theta}^2 + (fracI\lambda^2 sin(\theta)cos(\theta))v_x + (m_y - \frac{I}{\lambda^2} cos^2(\theta))v_y \end{bmatrix}$$

- where $\begin{bmatrix} v_x \\ v_y \end{bmatrix} =$

$$\begin{bmatrix} cos(\theta) & sin(\theta) \\ sin(\theta) & -cos(\theta) \end{bmatrix} \begin{bmatrix} \frac{u_1}{cos(\theta)} + \lambda\dot{\theta}^2 \\ \lambda(u_2 cos^2(\theta) - 2\dot{theta}^2 tan(\theta) \end{bmatrix} ; , I = I_3 + m3l^2$$

- Therefore

$$\begin{aligned} \ddot{\varepsilon}_1 &= u_1 \\ \ddot{\varepsilon}_2 &= u_2 \\ \ddot{\varepsilon}_3 &= \varepsilon_2 u_1 \end{aligned}$$

- Objective: The states track the prescribed path by $\varepsilon_{ij}^{d}, \dot{\varepsilon}_{ij}^{d}$
- The reference trajectory will be stated by following dynamics:

$$\begin{aligned}\ddot{\varepsilon}_{11}^{d} &= u_{1d}\\ \ddot{\varepsilon}_{21}^{d} &= u_{2d}\\ \ddot{\varepsilon}_{31}^{d} &= \varepsilon_{21}^{d} u_{1d}\end{aligned}$$

- Define the tracking error $x = \varepsilon - \varepsilon_d$

$$\begin{aligned}\dot{x}_{11} &= x_{12} \quad \dot{x}_{12} = u_1 - u_{1d}\\ \dot{x}_{21} &= x_{22} \quad \dot{x}_{22} = u_2 - u_{2d}\\ \dot{x}_{31} &= x_{32} \quad \dot{x}_{32} = x_{21} u_{1d} + \varepsilon_{21}(u_1 - u_{1d})\end{aligned} \tag{8}$$

- Now the problem is finding u_1 and u_2 to make system (8) g.a.s.

- Let us redefine the system into three subsystems:

$$\Delta_1 \quad \begin{cases} \dot{x}_{31} = x_{32} \\ \dot{x}_{32} = x_{21}u_{1d} + \varepsilon_{21}(u_1 - u_{1d}) \end{cases}$$

$$\Delta_2 \quad \begin{cases} \dot{x}_{21} = x_{22} \\ \dot{x}_{22} = u_2 - u_{2d} \end{cases}$$

$$\Pi \quad \begin{cases} \dot{x}_{11} = x_{12} \\ \dot{x}_{12} = u_1 - u_{1d} \end{cases}$$

- First we find u_1 to stabilize $\Pi \rightsquigarrow u_1 = u_{1d}$
- Then find u_2 to stabilize Δ_1 and Δ_2

Stabilzing Π

- $\Pi \begin{cases} \dot{x}_{11} = x_{12} \\ \dot{x}_{12} = u_1 - u_{1d} \end{cases}$
- This system can be stabilized by defining

$$u_1 = u_{1d} - k_1 x_{11} - k_2 x_{12}, \quad k_1 > 0, \ k_2 > 0 \tag{9}$$

- where $P(\lambda) = \lambda^2 + k_1\lambda + k_2$ is Hurwitz

Stabilizing Δ_1

- Assuming that $u_1 - u_{1d} \equiv 0$, Δ_1 can be written as
$$\Delta_1 \begin{cases} \dot{x}_{31} = x_{32} \\ \dot{x}_{32} = x_{21} u_{1d} \end{cases}$$
- Objective looking to design a stabilizing feedback x_{21}
- Assume u_{1d} is uniformly bounded in t and smooth
- Considering x_{32} as virtual input
- It can be easily shown that $\phi_1 = -c_1 u_{1d}^2 x_{31}, \quad c_1 > 0$ can stabilize the first eq.
- Following the back stepping procedure $\rightsquigarrow$ stabilizing x_{21} is
$$x_{21} = \phi_2 = -\frac{1}{u_{d1}}[-(2c_1 \dot{u}_{1d} u_{1d} + 1)x_{13} - c_1 u_{1d}^2 x_{32} - c_2(c_1 u_{1d}^4 + u_{1d}^2 x_{32})] = -(c_1 c_2 u_{1d}^3 + 2c_1 \dot{u}_{1d} + u_{1d}^{-1})x_{13} - (c_1 u_{1d} + c_2 u_{1d})x_{32}$$

Stabilizing Δ_2

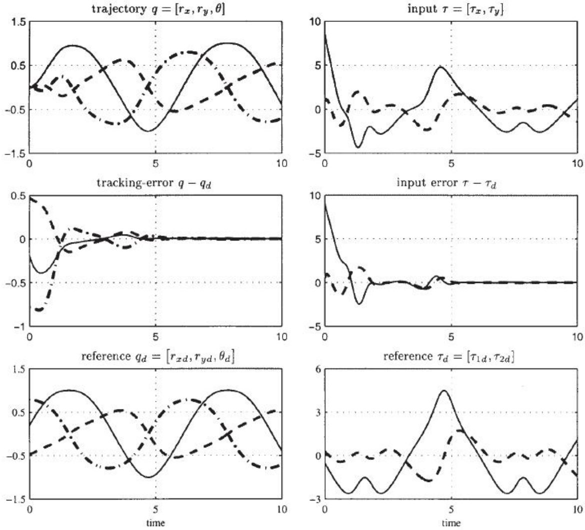
- $\Delta_2 \begin{cases} \dot{x}_{21} = x_{22} \\ \dot{x}_{22} = u_2 - u_{2d} \end{cases}$
- With $u_1 = u_{1d}$, Δ_1 is e.sabilized by $x_{21} = \phi_2$
- Apply backstepping to find u_2:
- Define $\bar{x}_{21} = x_{21} - \phi_2$
- $\therefore \dot{\bar{x}}_{21} = x_{22} - \frac{d}{dt}[\phi_2]$
- Now define $\bar{x}_{22} = x_{22} - \phi_3$, $\phi_3 = -c_3\bar{x}_{21} + \frac{d}{dt}[\phi_2]$
- It can be easily find that the following u_2 can stabilize the system

$$\begin{aligned} u_2 \quad - \quad u_{2d} &= -c_4\bar{x}_{22} + \frac{d}{dt}[\phi_3] \qquad (10) \\ &= -c_3c_4x_{21} - (c_3 + c_4)x_{22} + c_3c_4\phi_2 + (c_3 + c_4)\frac{d}{dt}[\phi_2] + \frac{d^2}{dt^2}[\phi_2] \end{aligned}$$

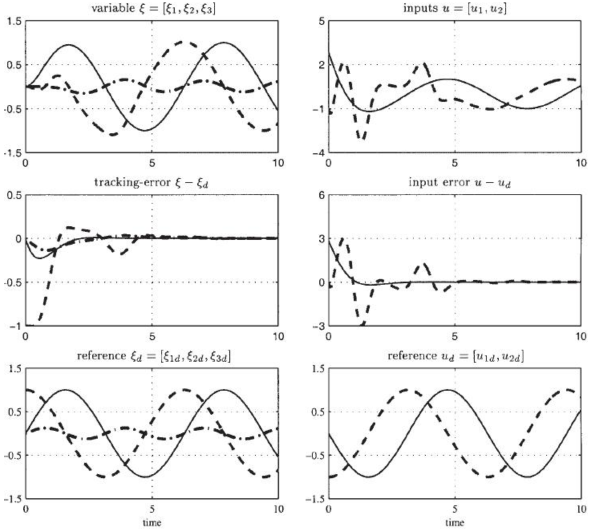
- It has been shown that (9) and (10) can e.stabilized (8). [1]

Simulation Results

- For the 3 link manipulator consider the following desired traj:
  $r_{xd} = r_1 sin(at) - \lambda(cos(arctan(r_2 cos(at))) - 1)$
  $r_{yd} = \frac{r_1 r_2}{8} sin(2at) - \lambda sin(arctan(r_2 cos(at)))$
  $\theta_d(t) = arctan(r_2 cos(at))$
- Define: $r_1 = r_2 = a = 1,\ k_1 = 4, k_2 = 2\sqrt{2}, c_1 = 2, c_2 = 2, c_3 = 4, c_4 = 4$
- The results of tracking is shown in Figs.



Tracking of the trajectory (37); co-ordinates of the mechanical system (27) with respect to time, r_x (solid), r_y (dashed), θ (dash–dotted), inputs τ_x (solid), τ_y (dashed).



Tracking of the trajectory (36); co-ordinates of the second-order chained form system (32), ξ_1 (solid), ξ_2 (dashed), ξ_3 (dash–dotted), inputs u_1 (solid) u_2 (dashed).

N. P. I. Aneke, H. Nijmeijerz, and A. G. de Jager, “Tracking control of second-order chained form systems by cascaded backstepping,” *Internaitonl Journal Of Robust And Nonlinear Control*, vol. 13, pp. 95–115, 2003.