



## Integrator Back Stepping

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# Integrator Back Stepping

- ▶ Let us start with integrator back stepping:

$$\dot{\eta} = f(\eta) + g(\eta)\varepsilon \quad (1)$$

$$\dot{\varepsilon} = u \quad (2)$$

- ▶  $[\eta^T \ \varepsilon]^T \in R^{n+1}$ : is the state
- ▶  $u \in R$ : control input
- ▶  $f : D \rightarrow R^n$  and  $g : D \rightarrow R^n$ : smooth in a domain  $D \subset R^n$ ;  $\eta = 0, f(0) = 0$
- ▶ **Objective:** Design a state FB controller to stabilize the origin ( $\eta = 0, \varepsilon = 0$ )
- ▶ We assume both  $f$  and  $g$  are known
- ▶ It is a cascade connection:
  - ▶ (1) with input  $\varepsilon$
  - ▶ Second is the integrator (2)

- Suppose (1) can be asym. stabilized

by  $\varepsilon = \phi(\eta)$  with  $\phi(0) = 0$ :

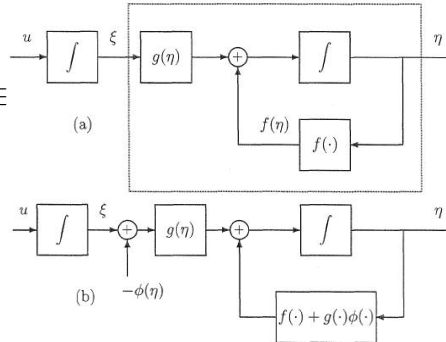
$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$$

- and  $V(\eta)$  is a smooth p.d. Lyap fcn:

$$\frac{\partial V}{\partial \eta} [f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta) \quad \forall \eta \in D, \quad W(\eta) \text{ is p.d.}$$

- Now add  $\pm g(\eta)\phi(\eta)$  to (1):

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] \\ &+ g(\eta)[\varepsilon - \phi(\eta)] \\ \dot{\varepsilon} &= u \end{aligned}$$



- Suppose (1) can be asym. stabilized

by  $\varepsilon = \phi(\eta)$  with  $\phi(0) = 0$ :

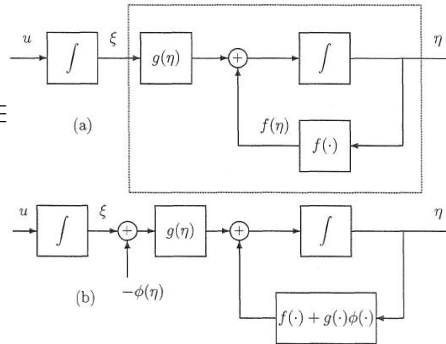
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- Now add  $\pm g(\eta)\phi(\eta)$  to (1):

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] \\ &+ g(\eta) \underbrace{[\varepsilon - \phi(\eta)]}_{\mathbf{z}} \end{aligned}$$



- Suppose (1) can be asym. stabilized

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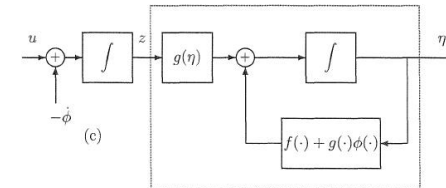
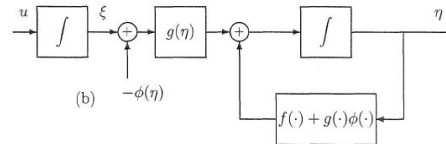
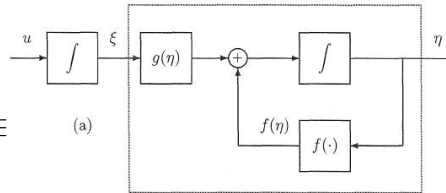
- Now add  $\pm g(\eta)\phi(\eta)$  to (1):

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] \\ &+ g(\eta) \underbrace{[\varepsilon - \phi(\eta)]}_{z} \end{aligned}$$

$$\therefore \dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z$$

$$\dot{z} = u - \dot{\phi}$$

$$\dot{\phi} = \frac{\partial \phi}{\partial \eta} [f(\eta) + g(\eta)\varepsilon]$$



► Fig b to Fig c is **back stepping**  $-\phi$  though the integrator

►  $v = u - \dot{\phi} \rightarrow$

$$\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z$$

$$\dot{z} = v$$

► It is similar to (1) But input zero  $\rightsquigarrow$  origin is a.s.

► Now let us design  $v$  to stabilize the over all system:

$$V_c(\eta, \varepsilon) = V(\eta) + \frac{1}{2}z^2$$

►  $\therefore \dot{V}_c \leq -W(\eta) + \frac{\partial V}{\partial \eta} g(\eta)z + zv$

► Choose  $v = -\frac{\partial V}{\partial \eta} g(\eta) - kz$ ,  $k > 0$

► So  $\dot{V}_c \leq -W(\eta) - kz^2$

►  $\therefore$  origin is a.s. ( $\eta = 0, z = 0$ )

►  $\phi(0) = 0 \rightsquigarrow \varepsilon = 0$

$$u = \frac{\partial \phi}{\partial \eta} [f(\eta) + g(\eta)\varepsilon] - \frac{\partial V}{\partial \eta} g(\eta) - k(\varepsilon - \phi(\eta)) \quad (3)$$

- **Lemma:** Consider the system (1)-(2). Let  $\phi(\eta)$  be a stabilizing state fb control law for (1) with  $\phi(0) = 0$ , and  $V(\eta)$  be a Lyap fcn that  $\dot{V} \leq -W(\eta)$  for some p.d fcn  $W(\eta)$ . Then, the state feedback control law (3) stabilizes the origin of (1)-(2), with  $V(\eta) + [\varepsilon - \phi(\eta)]^2/2$  as a Lyap fcn. Moreover, if all the assumptions hold globally and  $V(\eta)$  is "radially unbounded", the origin will be g.a.s.

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= u\end{aligned}$$

- Therefore  $\eta = x_1, \varepsilon = x_2$
- To stabilize  $x_1 = 0$ :  $x_2 = \phi(x_1) = -x_1^2 - x_1$
- $\therefore$  the nonlinear term  $x_1^2$  is canceled:  $\dot{x}_1 = -x_1 - x_1^3$ 
  - Why  $-x_1^3$  is not canceled?
- $V(x_1) = x_1^2/2 \rightsquigarrow \dot{V} = -x_1^2 - x_1^4 \leq -x_1^2, \forall x_1 \in R$
- $\therefore$  The origin of  $\dot{x}_1$  is g.e.s.



- To backstep:  $z_2 = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2$
- Hence

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_1^3 + z_2 \\ \dot{z}_2 &= u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)\end{aligned}$$

- Now take  $V_c = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$
- $\dot{V}_c = -x_1^2 - x_1^4 + z_2[x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + u]$
- $\therefore u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2 \rightsquigarrow \dot{V}_c = -x_1^2 - x_1^4 - z_2^2$
- The origin is g.a.s

- For higher order systems we can apply the recursive application of integrator back stepping
- **Example:** Consider

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u\end{aligned}$$

- After 1 back stepping:

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= x_3\end{aligned}$$

- that  $x_3$  is input is g.s. by:

$$x_3 = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - (x_2 + x_1 + x_1^2) = \phi(x_1, x_2)$$

- and  $V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2)^2$



## Back Stepping for More General Form

- Consider

$$\begin{aligned}\dot{\eta} &= f(\eta) + g(\eta)\varepsilon \\ \dot{\varepsilon} &= f_a(\eta, \varepsilon) + g_a(\eta, \varepsilon)u\end{aligned}\tag{4}$$

- $f_a$  and  $g_a$  are smooth
- If  $g_a(\eta, \varepsilon) \neq 0$  over the domain of interest: define

$$u = \frac{1}{g_a(\eta, \varepsilon)}[u_a - f_a(\eta, \varepsilon)]$$

- if a stabilizing state feedback control law  $\phi(\eta)$  and a Lyap fcn.  $V(\eta)$  exists s.t. satisfy the conditions of Lemma:

$$u = \frac{1}{g_a(\eta, \varepsilon)}\left[\frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\varepsilon] - \frac{\partial V}{\partial \eta}g(\eta) - k[\varepsilon - \phi(\eta)] - f_a(\eta, \varepsilon)\right] \quad k > 0$$

- and  $V_c(\eta, \varepsilon) = V(\eta) + \frac{1}{2}[\varepsilon - \phi(\eta)]^2$

# Back Stepping for Strict-Feedback Systems

- By recursive backstepping we strict-FB systems can be stabilized:

$$\begin{aligned}
 \dot{x} &= f_0(x) + g_0(x)z_1 \\
 \dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\
 \dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\
 &\vdots \\
 \dot{z}_{k-1} &= f_{k-1}(x, z_1, \dots, z_{k-1}) + g_{k-1}(x, z_1, \dots, z_{k-1})z_k \\
 \dot{z}_k &= f_k(x, z_1, \dots, z_k) + g_k(x, z_1, \dots, z_k)u
 \end{aligned}$$

- $x \in R^n$
- $z_1$  to  $z_k$  are scalar
- $f_0(0)$  to  $f_k(0)$  are zero
- $g_i(x, z_1, \dots, z_i) \neq 0$  for  $1 \leq i \leq k$  over the domain of interest
- "strict FB"  $\equiv f_i$  and  $g_i$  in  $\dot{z}_i$  **only** depends on  $x, z_1, \dots, z_i$

- Start the recursive procedure with  $\dot{x} = f_0(x) + g_0(x)z_1$
- Determine a stabilizing state fb  $z_1 = \phi_0(x)$ ,  $\phi_0(0) = 0$  and  $\frac{\partial V_0}{\partial x} [f_0(x) + g_0(x)\phi_0(x)] \leq -W(x)$ ,  $W(x)$  is p.d.
- Apply backstepping, consider

$$\begin{aligned}\dot{x} &= f_0(x) + g_0(x)z_1 \\ \dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2\end{aligned}$$

- The parameters can be defined as  
 $\eta = x, \quad \varepsilon = z_1, \quad u = z_2, \quad f = f_0, \quad g = g_0, \quad f_a = f_1, \quad g_a = g_1$
- The stabilizing state fb:  

$$\phi_1(x, z_1) = \frac{1}{g_1} \left[ \frac{\partial \phi_0}{\partial x} (f_0 + g_0 z_1) - \frac{\partial V_0}{\partial x} g_0 - k_1 (z_1 - \phi) - f_1 \right], \quad k_1 > 0$$
- The Lyap fcn:  $V_1(x, z_1) = V_0(x) + \frac{1}{2} [z_1 - \phi_1(x)]^2$

► Now consider:

$$\dot{x} = f_0(x) + g_0(x)z_1$$

$$\dot{z}_1 = f_1(x, z_1) + g_1(x, z_1)z_2$$

$$\dot{z}_2 = f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3$$

- The parameters can be defined as  $\eta = [x \ z_1]^T$ ,  $\varepsilon = z_2$ ,  $u = z_3$ ,  $f = [f_0 + g_0 z_1 \ f_1]^T$ ,  $g = [0 \ g_1]^T$ ,  $f_a = f_2$ ,  $g_a = g_2$
- The stabilizing state fb:  $\phi_2(x, z_1, z_2) = \frac{1}{g_2} \left[ \frac{\partial \phi_1}{\partial x} (f_0 + g_0 z_1) + \frac{\partial \phi_1}{\partial z_1} (f_1 + g_1 z_2) - \frac{\partial V_1}{\partial z_1} g_1 - k_2 (z_2 - \phi) - f_2 \right]$ ,  $k_2 > 0$
- The Lyap fcn:  $V_2(x, z_1, z_2) = V_1(x, z_1) + \frac{1}{2} [z_2 - \phi_2(x, z_1)]^2$
- This process should be repeated  $k$  times to obtain  $u = \phi_k(x, z_1, \dots, z_k)$  and Lyap fcn  $V_k(x, z_1, \dots, z_k)$
- **If a system has not defined in strict FB system, one can transform the states by normal transformation**

# Example

- Stabilize the following system

$$\begin{aligned}\dot{\eta} &= \eta^2 - \eta\varepsilon \\ \dot{\varepsilon} &= u\end{aligned}$$



## Example

- Stabilize the following system

$$\begin{aligned}\dot{\eta} &= \eta^2 - \eta\varepsilon \\ \dot{\varepsilon} &= u\end{aligned}$$

- By  $y = \varepsilon$  the system is nonminimum phase
- Define  $\varepsilon = \eta + \eta^2$  and  $V_0 = \frac{\eta^2}{2}$
- $\therefore \dot{V}_0 = -\eta^4, \forall \eta \in R$
- Use  $V = V_0 + (\varepsilon - \eta - \eta^2)^2/2 \rightsquigarrow \dot{V} = -\eta^4 + (\varepsilon - \eta - \eta^2)[- \eta^2 + u - (1 + 2\eta)(\eta^2 - \eta\varepsilon)]$
- so define  $u = \eta^2 + (1 + 2\eta)(\eta^2 - \eta\varepsilon) - k(\varepsilon - \eta - \eta^2), k > 0$
- $\therefore \dot{V} = -\eta^4 - k(\varepsilon - \eta - \eta^2)^2$

# Back Stepping for Uncertain Systems

- Consider the system:

$$\begin{aligned}\dot{\eta} &= f(\eta) + g(\eta)\varepsilon + \delta_{\eta}(\eta, \varepsilon) \\ \dot{\varepsilon} &= f_a(\eta, \varepsilon) + g_a(\eta, \varepsilon)u + \delta_{\varepsilon}(\eta, \varepsilon)\end{aligned}\tag{5}$$

- in domain  $D \subset R^{n+1}$
- contains  $(\eta = 0, \varepsilon = 0)$ ;  $\eta \in R^n, \varepsilon \in R$
- all fcns are smooth
- If  $g_a(\eta, \varepsilon) \neq 0$  over the domain of interest
- $f, g, f_a, g_a$  are known;  $\delta_{\eta}, \delta_{\varepsilon}$  are uncertain terms
- $f(0) = 0$  and  $f_a(0, 0) = 0$

$$\begin{aligned}\|\delta_{\eta}(\eta, \varepsilon)\|_2 &\leq a_1 \|\eta\|_2 \\ |\delta_{\varepsilon}(\eta, \varepsilon)| &\leq a_2 \|\eta\|_2 + a_3 |\varepsilon|\end{aligned}\tag{6}$$

- **Note:** The upper bound on  $\delta_{\eta}(\eta, \varepsilon)$  only depends on  $\eta$ .

- ▶  $V(\eta)$  for the first equation is  

$$\frac{\partial V}{\partial \eta} [f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \varepsilon)] \leq -b\|\eta\|_2^2; \text{ } b \text{ is pos. const.}$$
- ▶  $\therefore \dot{\eta} = 0$  is a.s. Equ. point of  $\dot{\eta} = f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \varepsilon)$
- ▶ Suppose  $|\phi(\eta)| \leq a_4\|\eta\|_2, \quad \left\| \frac{\partial \phi}{\partial \eta} \right\|_2 \leq a_5$  (7)
- ▶ Consider the Layp fcn for whole system:  

$$V_c(\eta, \varepsilon) = V(\eta) + \frac{1}{2}[\varepsilon - \phi(\eta)]^2$$
- ▶  $\dot{V}_c = \frac{\partial V}{\partial \eta} [f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \varepsilon)] + \frac{\partial V}{\partial \eta} g(\varepsilon - \phi) + (\varepsilon - \phi)[f_a + g_a u + \delta_\varepsilon - \frac{\partial \phi}{\partial \eta} (f + g\varepsilon + \delta_\eta)]$

- $V(\eta)$  for the first equation is  
 $\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \varepsilon)] \leq -b\|\eta\|_2^2$ ;  $b$  is pos. const.
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- Consider the Layp fcn for whole system:  
 $V_c(\eta, \varepsilon) = V(\eta) + \frac{1}{2}[\varepsilon - \phi(\eta)]^2$
- $\dot{V}_c = \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \varepsilon)] + \frac{\partial V}{\partial \eta}g(\varepsilon - \phi) + (\varepsilon - \phi)[f_a + g_a u + \delta_\varepsilon - \frac{\partial \phi}{\partial \eta}(f + g\varepsilon + \delta_\eta)]$
- Choose  $u = \frac{1}{g_a}[-f_a + \frac{\partial \phi}{\partial \eta}(f + g\varepsilon) - \frac{\partial V}{\partial \eta}g - k(\varepsilon - \phi)], \quad k > 0$
- $\dot{V}_c \leq -b\|\eta\|_2^2 + 2a_6\|\eta\|_2|\varepsilon - \phi| - (k - a_3)(\varepsilon - \phi)^2 =$   

$$- \begin{bmatrix} \|\eta\|_2 \\ |\varepsilon - \phi| \end{bmatrix}^T \underbrace{\begin{bmatrix} b & -a_6 \\ -a_6 & (k - a_3) \end{bmatrix}}_P \begin{bmatrix} \|\eta\|_2 \\ |\varepsilon - \phi| \end{bmatrix}; \quad a_6 = \frac{a_2 + a_5 a_1}{2}$$

- $V(\eta)$  for the first equation is  $\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \varepsilon)] \leq -b\|\eta\|_2^2$ ;  $b$  is pos. const.
- $\therefore \dot{\eta} = 0$  is a.s. Equ. point of  $\dot{\eta} = f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \varepsilon)$
- Suppose  $|\phi(\eta)| \leq a_4\|\eta\|_2, \quad \|\frac{\partial \phi}{\partial \eta}\|_2 \leq a_5$  (7)
- Consider the Layp fcn for whole system:  
 $V_c(\eta, \varepsilon) = V(\eta) + \frac{1}{2}[\varepsilon - \phi(\eta)]^2$
- $\dot{V}_c = \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \varepsilon)] + \frac{\partial V}{\partial \eta}g(\varepsilon - \phi) + (\varepsilon - \phi)[f_a + g_a u + \delta_\varepsilon - \frac{\partial \phi}{\partial \eta}(f + g\varepsilon + \delta_\eta)]$
- Choose  $u = \frac{1}{g_a}[-f_a + \frac{\partial \phi}{\partial \eta}(f + g\varepsilon) - \frac{\partial V}{\partial \eta}g - k(\varepsilon - \phi)], \quad k > 0$
- $\dot{V}_c \leq -b\|\eta\|_2^2 + 2a_6\|\eta\|_2|\varepsilon - \phi| - (k - a_3)(\varepsilon - \phi)^2 =$   

$$- \begin{bmatrix} \|\eta\|_2 \\ |\varepsilon - \phi| \end{bmatrix}^T \underbrace{\begin{bmatrix} b & -a_6 \\ -a_6 & (k - a_3) \end{bmatrix}}_P \begin{bmatrix} \|\eta\|_2 \\ |\varepsilon - \phi| \end{bmatrix}; \quad a_6 = \frac{a_2 + a_5 a_1}{2}$$
- Choose  $k > a_3 + \frac{a_6^2}{b} \rightsquigarrow \dot{V}_c \leq -\lambda_{\min}(P)[\|\eta\|_2^2 + |\varepsilon - \phi|^2]$

- **Lemma:** Consider the system (5), where the uncertainty satisfies inequalities (6). Let  $\phi(\eta)$  be a stabilizing state fb control law that satisfies (7), and  $V(\eta)$  be a Lyap. fcn that guarantee a.s. of the first Equatin of (5). Then the given state feedback control law in previous slide, with  $k$  sufficiently large, stabilizes the origin of (5). Moreover, if all the assumptions hold globally and  $V(\eta)$  is radially unbounded, the origin will be g.a.s.

# Trajectory Tracking for A Second Order System [1], [2]

- ▶ A 3 link underactuated manipulator
  - ▶ The first two translational joints are actuated
  - ▶ The third revolute joint is not actuated
  - ▶ The linear approximation of this system is not controllable since it is not influenced by gravity

- ▶ The dynamics:

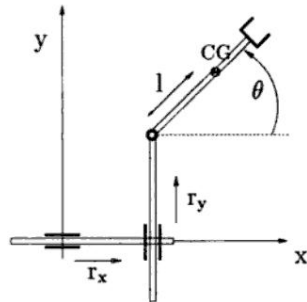
$$m_x \ddot{r}_x - m_3 l \sin(\theta) \dot{\theta} - m_3 l \cos(\theta) \dot{\theta}^2 = \tau_1$$

$$m_y \ddot{r}_y + m_3 l \cos(\theta) \dot{\theta} - m_3 l \sin(\theta) \dot{\theta}^2 = \tau_2$$

$$I \ddot{\theta} - m_3 l \sin(\theta) \ddot{r}_x + m_3 l \cos(\theta) \ddot{r}_y = 0$$

$$\lambda \ddot{\theta} + \ddot{r}_x \sin(\theta) + \ddot{r}_y \cos(\theta) = 0$$

where  $[r_x, r_y]$ : displacement of third joint;  $\theta$  orientation of third link respect to x axis;  $\tau_1 \tau_2$ : input of actuated joints;  $m_i$ : mass,  $I_i$ : inertia;  $\lambda = (I_3 + m_3 l^2)/(m_3 l)$



- Transform the dynamics by:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} r_x + \lambda(\cos(\theta) - 1) \\ \tan(\theta) \\ r_y + \lambda\sin(\theta) \end{bmatrix}; \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -m_3 l \cos(\theta) \dot{\theta}^2 + (m_x - \frac{l}{\lambda^2} \sin^2(\theta)) v_x + (\frac{l}{\lambda^2} \sin(\theta) \cos(\theta)) v_y \\ -m_3 l \sin(\theta) \dot{\theta}^2 + (\frac{l}{\lambda^2} \sin^2(\theta) \cos(\theta)) v_x + (m_y - \frac{l}{\lambda^2} \cos^2(\theta)) v_y \end{bmatrix}$$

- where  $\begin{bmatrix} v_x \\ v_y \end{bmatrix} =$

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} \frac{u_1}{\cos(\theta)} + \lambda \dot{\theta}^2 \\ \lambda(u_2 \cos^2(\theta) - 2\theta \dot{\theta}^2 \tan(\theta)) \end{bmatrix}; I = I_3 + m3l^2$$

- Therefore

$$\ddot{\varepsilon}_1 = u_1$$

$$\ddot{\varepsilon}_2 = u_2$$

$$\ddot{\epsilon}_3 = \epsilon_2 u_1$$



- **Objective:** The states track the prescribed path by  $\varepsilon_{ij}^d, \dot{\varepsilon}_{ij}^d$
- Define the tracking error  $x = \varepsilon - \varepsilon_d$

$$\begin{aligned}\dot{x}_{11} &= x_{12} & \dot{x}_{12} &= u_1 - u_{1d} \\ \dot{x}_{21} &= x_{22} & \dot{x}_{21} &= u_2 - u_{2d} \\ \dot{x}_{31} &= x_{32} & \dot{x}_{32} &= x_{21}u_{1d} + \varepsilon_{21}(u_1 - u_{1d})\end{aligned}\tag{8}$$

- Now the problem is finding  $u_1$  and  $u_2$  to make system (8) g.a.s.

- Let us redefine the system into three subsystems:

$$\begin{aligned}\Delta_1 & \begin{cases} \dot{x}_{31} = x_{32} \\ \dot{x}_{32} = x_{21}u_{1d} + \varepsilon_{21}(u_1 - u_{1d}) \end{cases} \\ \Delta_2 & \begin{cases} \dot{x}_{21} = x_{22} \\ \dot{x}_{22} = u_2 - u_{2d} \end{cases} \\ \Pi & \begin{cases} \dot{x}_{11} = x_{12} \\ \dot{x}_{12} = u_1 - u_{1d} \end{cases}\end{aligned}$$

- First we find  $u_1$  to stabilize  $\Pi \rightsquigarrow u_1 = u_{1d}$
- Then find  $u_2$  to stabilize  $\Delta_1$  and  $\Delta_2$

# Stabilizing $\Pi$

- ▶  $\Pi \begin{cases} \dot{x}_{11} = x_{12} \\ \dot{x}_{12} = u_1 - u_{1d} \end{cases}$
- ▶ This system can be stabilized by defining

$$u_1 = u_{1d} - k_1 x_{11} - k_2 x_{12}, \quad k_1 > 0, \quad k_2 > 0 \quad (9)$$

- ▶ where  $P(\lambda) = \lambda^2 + k_1 \lambda + k_2$  is Hurwitz

## Stabilizing $\Delta_1$

- Assuming that  $u_1 - u_{1d} \equiv 0$ ,  $\Delta_1$  can be written as

$$\Delta_1 \begin{cases} \dot{x}_{31} = x_{32} \\ \dot{x}_{32} = x_{21} u_{1d} \end{cases}$$

- Objective** looking to design a stabilizing feedback  $x_{21}$
- Assume  $u_{1d}$  is uniformly bounded in  $t$  and smooth
- Considering  $x_{32}$  as virtual input
- It can be easily shown that  $\phi_1 = -c_1 u_{1d}^2 x_{31}$ ,  $c_1 > 0$  can stabilize the first eq.
- Following the back stepping procedure  $\rightsquigarrow$  stabilizing  $x_{21}$  is  

$$x_{21} = \phi_2 = -\frac{1}{u_{1d}} [-(2c_1 \dot{u}_{1d} u_{1d} + 1)x_{31} - c_1 u_{1d}^2 x_{32} - c_2 (c_1 u_{1d}^4 + u_{1d}^2 x_{32})] =$$

$$-(c_1 c_2 u_{1d}^3 + 2c_1 \dot{u}_{1d} + u_{1d}^{-1})x_{31} - (c_1 u_{1d} + c_2 u_{1d})x_{32}$$

## Stabilizing $\Delta_2$

- ▶  $\Delta_2 \begin{cases} \dot{x}_{21} = x_{22} \\ \dot{x}_{22} = u_2 - u_{2d} \end{cases}$
- ▶ With  $u_1 = u_{1d}$ ,  $\Delta_1$  is e.stabilized by  $x_{12} = \phi_2$
- ▶ Apply backstepping to find  $u_2$ :
- ▶ Define  $\bar{x}_{21} = x_{21} - \phi_2$
- ▶  $\therefore \dot{\bar{x}}_{21} = x_{21} - \frac{d}{dt}[\phi_2]$
- ▶ Now define  $\bar{x}_{22} = x_{22} - \phi_3$ ,  $\phi_3 = -c_3\bar{x}_{21} + \frac{d}{dt}[\phi_2]$
- ▶ It can be easily find that the following  $u_2$  can stabilize the system

$$u_2 - u_{2d} = -c_4\bar{x}_{22} + \frac{d}{dt}[\phi_3] \quad (10)$$

$$= -c_3c_4x_{21} - (c_3 + c_4)x_{22} + c_3c_4\phi_2 + (c_3 + c_4)\frac{d}{dt}[\phi_2] + \frac{d^2}{dt^2}[\phi_2]$$

- ▶ It has been shown that (9) and (10) can e.stabilized (8). [1]

# Simulation Results

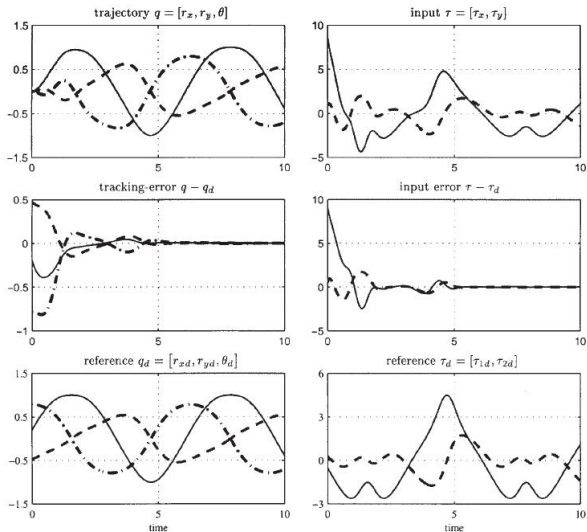
- For the 3 link manipulator consider the following desired traj:

$$r_{xd} = r_1 \sin(at) - \lambda(\cos(\arctan(r_2 \cos(at))) - 1)$$

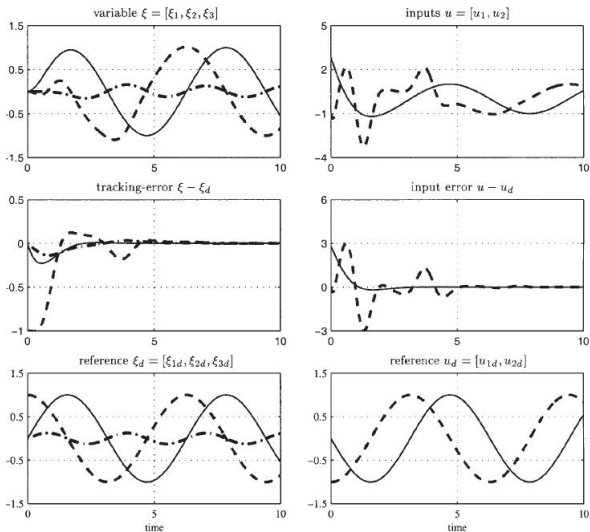
$$r_{yd} = \frac{r_1 r_2}{8} \sin(2at) - \lambda \sin(\arctan(r_2 \cos(at)))$$

$$\theta_d(t) = \arctan(r_2 \cos(at))$$

- Define:  $r_1 = r_2 = a = 1$ ,  $k_1 = 4$ ,  $k_2 = 2\sqrt{2}$ ,  $c_1 = 2$ ,  $c_2 = 2$ ,  $c_3 = 4$ ,  $c_4 = 4$
- The results of tracking is shown in Figs.



Tracking of the trajectory (37); co-ordinates of the mechanical system (27) with respect to time,  $r_x$  (solid),  $r_y$  (dashed),  $\theta$  (dash-dotted), inputs  $\tau_x$  (solid),  $\tau_y$  (dashed).



Tracking of the trajectory (36); co-ordinates of the second-order chained form system (32),  $\xi_1$  (solid),  $\xi_2$  (dashed),  $\xi_3$  (dash-dotted), inputs  $u_1$  (solid)  $u_2$  (dashed).





N. P. Aneke, y. H. Nijmeijerz, and A. G. Jager, “Tracking control of second-order chained form systems by cascaded backstepping,” *International Journal of Robust and Nonlinear Control* vol. 95, no.115 , pp. 95–115, 2003.



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Springer, 2001.