

Computational Intelligence

Lecture 10:Fuzzy Sets

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Classical Set

Fuzzy Set

- Basic Concepts in Fuzzy Sets

- Operations on Fuzzy Sets

- Fuzzy Complement

- Fuzzy Union

- Fuzzy Intersection

- Averaging Operator

Classical Set

- ▶ A classical (crisp) set A in the universe of discourse U : can be defined by
 - ▶ **List method**: listing all of its members
 - ▶ **Rule method**: specifying the properties that must be satisfied by the members of the set

$$A = \{x \in U \mid x \text{ meets some conditions}\}$$

- ▶ **Membership method**: introduces a zero-one membership function (also called characteristic function, discrimination function, or indicator function)

$$\mu_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Example: cars in Tehran

- ▶ The universe of discourse U .
- ▶ Set A is the cars with 4 cylinders:

$$A = \{x \in U \mid x \text{ has 4 cylinders}\} \text{ OR}$$

$$\mu_A = \begin{cases} 1 & \text{if } x \in U \text{ \& } x \text{ has 4 cylinders} \\ 0 & \text{if } x \in U \text{ \& } x \text{ does not have 4 cylinder} \end{cases}$$



- ▶ Set D is the car made in Iran
- ▶ BUT the distinction between an Iranian car and a non-Iranian a car is not crisp:(
 - ▶ Most of them are not completely made in Iran
- ▶ So what should we do??!!

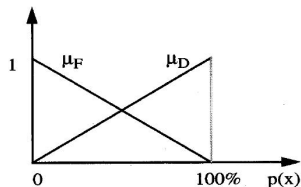
Fuzzy Set

- ▶ some sets do not have clear boundaries.
- ▶ **Fuzzy set:** in a universe of discourse U is characterized by a membership function $\mu_A(x)$ that takes values in the interval $[0, 1]$.
- ▶ In classical sets the membership function of a classical set can only take **zero and one**
- ▶ In fuzzy set the membership function is a **continuous function** with range $[0, 1]$.
- ▶ A fuzzy set A in U is represented by:
 - ▶ a set of ordered pairs of a generic element x and its membership value:

$$A = \{(x, \mu_A(x)) | x \in U\}$$
 - ▶ for **continuous** U : $A = \int_U \mu_A(x) / x$.
 - ▶ for **discrete** U : $\mu_A(x)$: $A = \sum_U \mu_A(x) / x$
 - ▶ \int and \sum do not represent integral and summation.
 - ▶ They denote collection of all points $x \in U$ with the associated membership function $\mu_A(x)$

Example: cars in Tehran (Cont'd)

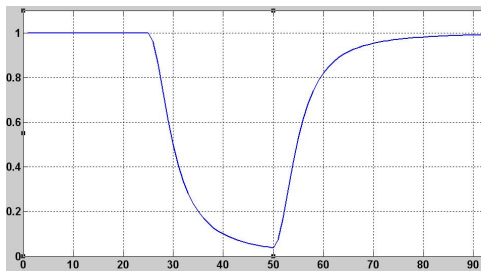
- ▶ D : The set "Iranian cars in Iran,"
- ▶ $\mu_D = p(x)$
 - ▶ $p(x)$ is the percentage of the parts of car x made in Iran
 - ▶ it takes values from 0% to 100%.
- ▶ F : The set "non-Iranian cars in Iran,"
- ▶ $\mu_F(x) = 1 - p(x)$



- ▶ Different membership functions can be defined to characterize the same description.
- ▶ The membership functions are not fuzzy, themselves.
- ▶ They are precise mathematical functions.
- ▶ Fuzzy sets are used to defuzzify the world.
- ▶ How to determine the membership functions?
 - ▶ Formulate human knowledge
 - ▶ Usually, gives a rough formula of the membership function
 - ▶ fine-tuning is required.
 - ▶ Data collected from various sensors
 - ▶ specify the structures of the membership functions and then fine-tune the parameters based on the data.
- ▶ A fuzzy set has a one-to-one correspondence with its membership function

Example: Old and Young [?]

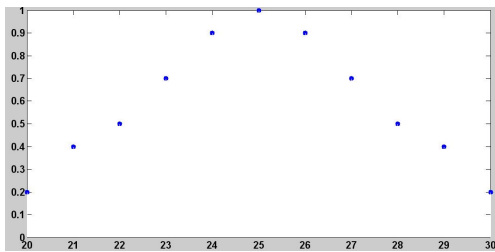
- ▶ U is in the interval of $[0, 100]$
- ▶ $young = \int_0^{25} 1/x + \int_{25}^{100} (1 + (\frac{x-25}{5})^2)^{-1}/x$
- ▶ $old = \int_{50}^{100} (1 + (\frac{x-50}{5})^{-2})^{-1}/x$



Example: A Digital Thermometer

- ▶ T : the set for desirable temperature
- ▶ $U \in [18, 33]$

$$\begin{aligned} \mu_T &= \frac{0}{20} + \frac{.4}{21} + \frac{.5}{22} + \frac{.7}{23} \\ &+ \frac{.9}{24} + \frac{1}{25} + \frac{.9}{26} + \frac{.7}{27} \\ &+ \frac{.5}{28} + \frac{.4}{29} + \frac{.2}{30} \end{aligned}$$



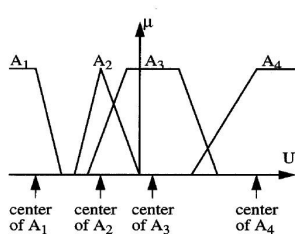
Basic Concepts in Fuzzy Sets

- ▶ **Support of a fuzzy set A** in the universe of discourse U is a crisp set that contains all the elements of U that have **nonzero membership values** in A : $supp_A = \{x \in U | \mu_A > 0\}$

- ▶ In the digital thermometer example: $supp_A = [21, 30]$
- ▶ **empty fuzzy set**: support is empty
- ▶ **fuzzy singleton**: support is a single point

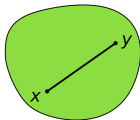
- ▶ **Center of a fuzzy set**:

- ▶ If the **mean value** of all points at which the membership function of the fuzzy set achieves its maximum value is **finite**, then this mean value is the center
- ▶ If the **mean value** equals positive (negative) **infinite**, then the center is the smallest (largest) among all points that achieve the maximum membership value.



- ▶ **Crossover point of a fuzzy set:** the point in U whose membership value in A equals 0.5.
- ▶ **Height of a fuzzy set:** the largest membership value attained by any point.
 - ▶ **Normal fuzzy set:** the height of fuzzy set equals to one (digital thermometer).
- ▶ **α -cut of a fuzzy set A** a crisp set A_α contains all the elements in U that have membership values in A greater than or equal to α :
$$A_\alpha = \{x \in U \mid \mu_A(x) \geq \alpha\}$$
 - ▶ In digital thermometer for $\alpha = 0.7$, $T_\alpha = [23, 24, 25, 26, 27]$
 - ▶ A fuzzy set A is **convex** iff its α -cut is a convex set for $\forall \alpha \in (0, 1]$.

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 - ▶ A fuzzy set A is **convex** iff its α -cut is a convex set for $\forall \alpha \in (0, 1]$.
 - ▶ In Euclidean space, an object is **convex** if for every pair of points within the object, every point on the **straight line segment** that joins them is also within the object.



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 - ▶ A fuzzy set A is **convex** iff its α -cut is a convex set for $\forall \alpha \in (0, 1]$.
 - ▶ Let C be a set in a real or complex vector space. C is convex if, $\forall x, y \in C$ and all $\lambda \in [0, 1] \rightsquigarrow \lambda x + (1 - \lambda)y \in C$

- ▶ **Crossover point of a fuzzy set:** the point in U whose membership value in A equals 0.5.
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 - ▶ In digital thermometer for $\alpha = 0.7$, $T_\alpha = [23, 24, 25, 26, 27]$
 - ▶ A fuzzy set A is **convex** iff its α -cut is a convex set for $\forall \alpha \in (0, 1]$.
 - ▶ **Lemma:** A fuzzy set $A \in \mathcal{R}^n$ is convex iff

$$\mu_A[\lambda x_1 + (1 - \lambda)x_2] \geq \min[\mu_A(x_1), \mu_A(x_2)] \quad \forall x_1, x_2 \in \mathcal{R}^n, \lambda \in [0, 1].$$

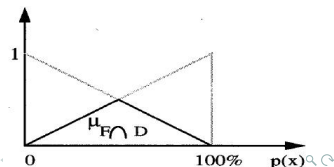
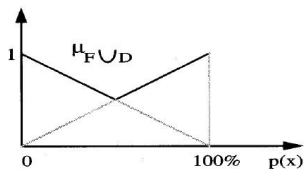
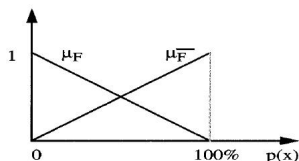
Operations on Fuzzy Sets

- ▶ Sets F and D are **equal** iff

$$\mu_F(x) = \mu_D(x), \forall x \in U$$
- ▶ Set D **contains** set F ($F \subset D$), iff

$$\mu_F(x) \leq \mu_D(x), \forall x \in U$$
- ▶ **Complement of F** is a fuzzy set $\bar{F} \in U$ whose membership function is

$$\mu_{\bar{F}}(x) = 1 - \mu_F(x)$$
- ▶ **Union of sets F and D ($F \cup D$)** is a fuzzy set in U : $\mu_{F \cup D} = \max[\mu_F(x), \mu_D(x)]$
 - ▶ $F \cup D$ is the smallest fuzzy set containing both F and D .
- ▶ **Intersection of F and D ($F \cap D$)** is a fuzzy set in U : $\mu_{F \cap D} = \min[\mu_F(x), \mu_D(x)]$
 - ▶ $F \cap D$ is the smallest fuzzy set contained by F and D .



- ▶ The De Morgan's Laws are true for fuzzy sets:

$$\overline{F \cup D} = \bar{F} \cap \bar{D}$$

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- ▶ For Iranian Cars example:

- ▶ $\mu_{F \cup D} = \begin{cases} \mu_D & \text{if } 0 \leq p(x) \leq 0.5 \\ \mu_F & \text{if } 0.5 \leq p(x) \leq 1 \end{cases}$
- ▶ $\mu_{F \cap D} = \begin{cases} \mu_F & \text{if } 0 \leq p(x) \leq 0.5 \\ \mu_D & \text{if } 0.5 \leq p(x) \leq 1 \end{cases}$

Further Operations

- ▶ An other difference between fuzzy sets and crisp sets:
 - ▶ for crisp sets only one type of operation is defined for complement, union, and intersection
 - ▶ for fuzzy sets, we can define several operations for them based on the given axioms.
- ▶ Why do we need different type of operations?
 - ▶ Some operations may not be satisfactory in some situations.

Fuzzy Complement

- ▶ Let $c : [0, 1] \rightarrow [0, 1]$ be a mapping that transforms the membership function of fuzzy set A into the membership function of the **complement of A** : $c[\mu_A(x)] = \mu_{\bar{A}}(x)$
- ▶ It was defined: $c[\mu_A(x)] = 1 - \mu_A$
- ▶ Let $a = \mu_A(x_1)$ and $b = \mu_A(x_2)$
- ▶ the function c is qualified as a complement if:
 - ▶ **Axiom c1**: $c(0) = 1$ and $c(1) = 0$ (boundary condition)
 - ▶ **Axiom c2**: $\forall a, b \in [0, 1]$, if $a < b$, then $c(a) \geq c(b)$ (nonincreasing condition)
 - ▶ an increase in membership value must result in a decrease or no change in membership value for the complement

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 - ▶ an increase in membership value must result in a decrease or no change in membership value for the complement
- ▶ Some types of fuzzy complement:
 - ▶ Sugeno class: $c_\lambda(a) = \frac{1-a}{1+\lambda a}$, $\lambda \in (-1, \infty)$
 - ▶ $\lambda = 0 \rightsquigarrow$ basic fuzzy complement
 - ▶ Yager class: $c_w(a) = (1 - a^w)^{1/w}$, $w \in (0, \infty)$
 - ▶ $w = 1 \rightsquigarrow$ basic fuzzy complement

Fuzzy set-S Norm

- ▶ Let $s : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a mapping that transforms the membership functions of fuzzy sets A and B into the membership function of the union of A and B , that $s[\mu_A(x), \mu_B(x)] = \mu_{A \cup B}$.
- ▶ the function S to be qualified as an union
- ▶ Let $a = \mu_A(x)$ and $b = \mu_B(x)$
 - ▶ **Axiom s1.** $s(1, 1) = 1, s(0, a) = s(a, 0) = a$ (boundary condition).
 - ▶ **Axiom s2.** $s(a, b) = s(b, a)$ (commutative condition).
 - ▶ **Axiom s3.** If $a \leq a'$ and $b \leq b'$, then $s(a, b) \leq s(a', b')$ (nondecreasing condition).
 - ▶ **Axiom s4.** $s(s(a, b), c) = s(a, s(b, c))$ (associative condition).
- ▶ Popular types of s-norm
 - ▶ Dombi class: $s_\lambda(a, b) = \frac{1}{1 + [(\frac{1}{a} - 1)^{-\lambda} + (\frac{1}{b} - 1)^{-\lambda}]^{-1/\lambda}}$, $\lambda \in (0, \infty)$
 - ▶ Dobios-Prade class: $s_\alpha(a, b) = \frac{a+b-ab-\min(a,b,1-\alpha)}{\max(1-a,1-b,\alpha)}$, $\alpha \in [0, 1]$
 - ▶ Yager class: $s_w(a, b) = \min[1, (a^w + b^w)^{1/w}]$, $w \in (0, \infty)$

► Other type of s-norm

- Drastic sum: $s_{ds}(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$
- Einstein sum: $s_{es}(a, b) = \frac{a+b}{1+ab}$
- Algebraic sum: $s_{as}(a, b) = a + b - ab$

► **Theorem:** For any s-norm s , that is for any function $s : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that satisfies Axioms s1-s4, the smallest s-norm is maximum and the largest is drastic s-norm

► Proof:

- Axioms s1 and s3 $\Rightarrow s(a, b) \geq s(a, 0) = a$
- Axiom s2 $\Rightarrow s(a, b) = s(b, a) \geq s(b, 0) = b$
- $\therefore s(a, b) \geq \max(a, b)$
- If $b = 0$, Axiom s1 $\Rightarrow s(a, b) = s(a, 0) = a \rightsquigarrow s(a, b) = s_{ds}(a, b)$
- If $a = 0$, Axiom s2 $\Rightarrow s(a, b) = s_{ds}(a, b)$
- If $a \neq 0 \& b \neq 0$, $s_{ds}(a, b) = 1 \geq s(a, b)$
- $\therefore s(a, b) \leq s_{ds}(a, b), \forall a, b \in [0, 1]$

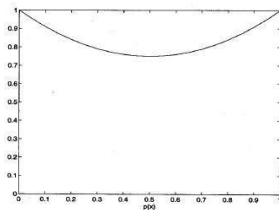
▶ **Example:** The Iranian cars

- ▶ Using Algebraic sum:

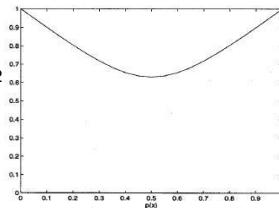
$$\mu_{F \cup D} = p(x) + (1 - p(x)) - p(x)(1 - p(x)) = 1 - p(x) + p(x)^2$$

- ▶ Using Yager s-norm, $w = 3$:

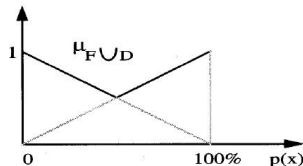
$$\mu_{F \cup D} = \min[1, (p(x)^3 + (1 - p(x))^3)^{1/3}]$$



Algebraic



Yager



max

Classical Set

- **Lemma 1:** For Dombi class s-norm and Drastic class s-norm it can be defined

$$\lim_{\lambda \rightarrow \infty} s_{\lambda}(a, b) = \max(a, b)$$

$$\lim_{\lambda \rightarrow 0} s_{\lambda}(a, b) = s_{ds}(a, b)$$

- **Lemma 2:** For Yager class s-norm and Drastic class s-norm it can be defined

$$\lim_{w \rightarrow \infty} s_w(a, b) = \max(a, b)$$

$$\lim_{w \rightarrow 0} s_w(a, b) = s_{ds}(a, b)$$

Fuzzy Intersection- T-Norm

- ▶ Let $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a mapping that transforms the membership functions of fuzzy sets A and B into the membership function of the union of A and B , that $t[\mu_A(x), \mu_B(x)] = \mu_{A \cap B}$.
- ▶ the function T to be qualified as an intersection
 - ▶ **Axiom t1.** $t(0, 0) = 0, t(1, a) = t(a, 1) = 1$ (boundary condition).
 - ▶ **Axiom t2.** $t(a, b) = t(b, a)$ (commutative condition).
 - ▶ **Axiom t3.** If $a \leq a'$ and $b \leq b'$, then $t(a, b) \leq t(a', b')$ (nondecreasing condition).
 - ▶ **Axiom t4.** $t(t(a, b), c) = t(a, t(b, c))$ (associative condition).

▶ Popular types of t -norm

- ▶ Dombi class: $t_\lambda(a, b) = \frac{1}{1 + [(\frac{1}{a} - 1)^\lambda + (\frac{1}{b} - 1)^\lambda]^{1/\lambda}}, \lambda \in (0, \infty)$
- ▶ Dobios-Prade class: $t_\alpha(a, b) = \frac{ab}{\max(a, b, \alpha)}, \alpha \in [0, 1]$
- ▶ Yager class:

$$t_w(a, b) = 1 - \min[1, ((1 - a)^w + (1 - b)^w)^{1/w}], w \in (0, \infty)$$

► Other type of t-norm

- Drastic product: $t_{ds}(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$
- Einstein product: $t_{ep}(a, b) = \frac{ab}{2-(a+b-ab)}$
- Algebraic product: $t_{ap}(a, b) = ab$

► **Theorem:** For any t-norm t , that is for any function $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that satisfies Axioms t1-t4, the largest t-norm is minimum and the smallest is drastic t-norm

► prove it.

► **Lemma 3:** For Dombi class t-norm and Drastic class t-norm it can be defined

$$\lim_{\lambda \rightarrow \infty} t_{\lambda}(a, b) = \min(a, b)$$

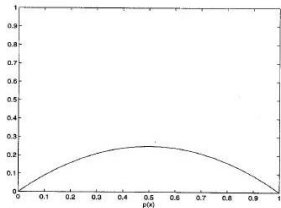
$$\lim_{\lambda \rightarrow 0} t_{\lambda}(a, b) = t_{dp}(a, b)$$



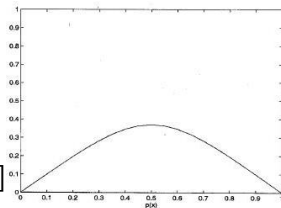
► **Example:** One more time, the Iranian cars

- Using Algebraic product:
- Using Yager t-norm, $w = 3$:

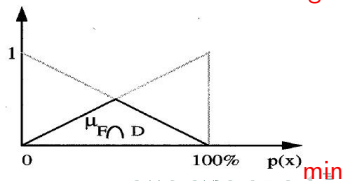
$$\mu_{F \cap D} = 1 - \min[1, ((1 - p(x))^3 + p(x)^3)^{1/3}]$$



Algebraic



Yager

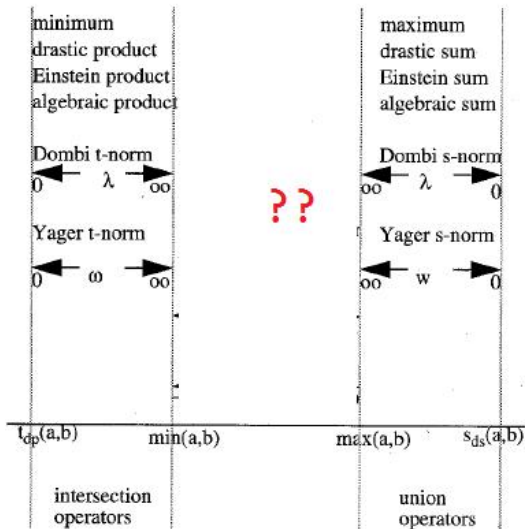


min

- ▶ If the s-norm $s(a, b)$, t-norm $t(a, b)$ and fuzzy complement $c(a)$ satisfy the following equation, they form an **associated class** (DeMorgan's Law)

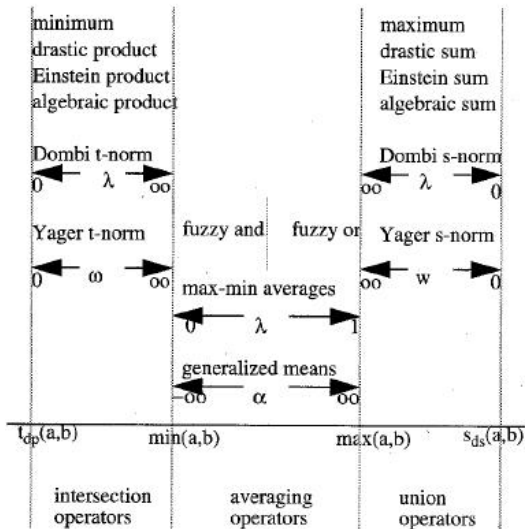
$$c[s(a, b)] = t[c(a), c(b)]$$

- ▶ **Example:** Show that the Yager s-norm and t-norm with the basic complement are associated
 - ▶ $c[s_w(a, b)] = 1 - \min[1, (a^w + b^w)^{1/w}]$
 - ▶ $t_w[c(a), c(b)] = 1 - \min[1, ((1 - 1 + a)^w + (1 - 1 + b)^w)^{1/w}]$



Averaging Operator

- ▶ This operator fills the gap between $\min(a, b)$, and $\max(a, b)$
- ▶ Some average operators:
 - ▶ Max-min average: $v_\lambda(a, b) = \lambda \max(a, b) + (1 - \lambda) \min(a, b)$, $\lambda \in [0, 1]$
 - ▶ Generalized means: $v_\alpha(a, b) = \left(\frac{a^\alpha + b^\alpha}{2}\right)^{1/\alpha}$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$
 - ▶ Fuzzy and: $v_p(a, b) = p \min(a, b) + \frac{(1-p)(a+b)}{2}$, $p \in [0, 1]$
 - ▶ Fuzzy or: $v_\gamma(a, b) = \gamma \max(a, b) + \frac{(1-\gamma)(a+b)}{2}$, $\gamma \in [0, 1]$



Full Scope of Fuzzy Operators