Computational Intelligence
Lecture 10:Fuzzy Relations

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Classical Relation

Fuzzy Relation

- Projection
- Cylindrical Extension
- Cartesian Product of Fuzzy Sets
- Composition

Extension Principle

- **Cartesian Product ($U_1 \times U_2 \times \ldots \times U_n$):**
  - U_i $t = 1, ..., n$: n arbitrary classical sets.
  - $U_1 \times U_2 \times \ldots \times U_n$ is the set of all ordered n-tuples $(u_1, ..., u_n)$:
    $U_1 \times U_2 \times \ldots \times U_n = \{(u_1, u_2, ..., u_n) | u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n\}$
- For binary relation ($n = 2$): $U_1 \times U_2 = \{(u_1, u_2) | u_1 \in U_1, u_2 \in U_2\}$
- $U_1 \neq U_2 \rightsquigarrow U_1 \times U_2 \neq U_2 \times U_1$.
- **A relation among sets $U_1, U_2, \ldots, U_n$ ($Q(U_1, U_2, \ldots, U_n)$)**:
  - a subset of the Cartesian product $U_1 \times U_2 \times \ldots \times U_n$:
    $Q(U_1, U_2, \ldots, U_n) \subset U_1 \times U_2 \times \ldots \times U_n$
- a relation is itself a set $\rightsquigarrow$, all of the basic set operations can be applied to it without modification.
- It can be represented by membership function:

$$\mu_Q(u_1, \ldots, u_n) = \begin{cases} 1 & if \ (u_1, \ldots, u_n) \in Q(U_1, U_2, \ldots, U_n) \\ 0 & otherwise \end{cases}$$

- The values of the membership function μ_Q can be shown by a relational matrix.

Example:

- $U = \{1, 2, 3\},\ \ V = \{a, b\}$
- $U \times V = (1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
- Let $Q(U, V)$ be a relation named "the first element is not smaller than 2"
- $Q(U, V) = \{(2, a), (2, b), (3, a), (3, b)\}$

$U \diagdown V$	a	b
1	0	0
2	1	1
3	1	1

Fuzzy Relation

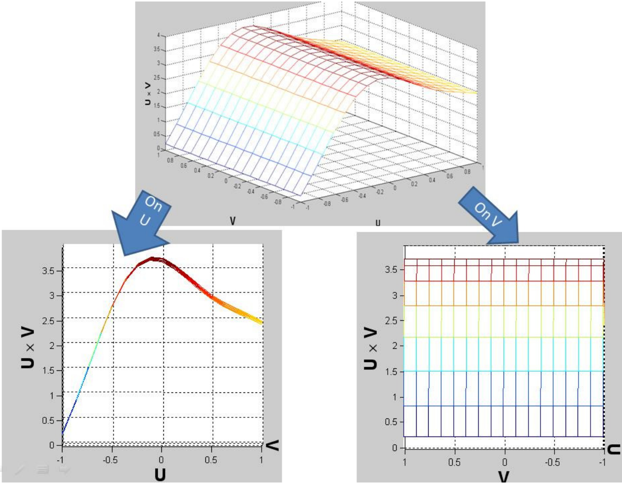
- A classical relation represents a crisp (zero-one) relationship among sets.
- But, for certain relationships, it is difficult to express the relation by a zero-one assessment
- In fuzzy relation the degree the strength of the relation is defined by different membership on the unit interval $[0, 1]$.
- **A fuzzy relation** is a fuzzy set defined in the Cartesian product of crisp sets $U_1, U_2, ..., U_n$.
  $Q = \{((u_1, u_2, ..., u_n), \mu_Q(u_1, u_2, ..., u_n))|(u_1, u_2, ..., u_n) \in U_1 \times U_2 \times \ldots, \times U_n\}, \quad \mu_Q : U_1 \times U_2 \times \ldots, \times U_n \to [0, 1]$
- Example:Fuzzy relation: "x is approximately equal to y" (AE).
  - $U = V = R$,
  - $\mu_{AE}(x, y) = e^{-(x-y)^2}$
  - This membership function is not unique

Example: Dormitory based on Distance of cities.

- $V = \{Tehran, Tabriz, Karaj, Qom\}, U = \{Tehran, Esfahan\}$
- Relation: "very far"
- use number between 0 and 1 for degree of relation

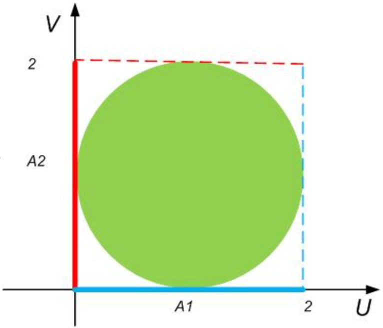
U / V	Tehran	Tabriz	Karaj	Qom
Tehran	0	0.9	0.1	0.3
Esfahan	0.7	0.95	0.8	0.5

Projection



Projection

- **Example:** A crisp relation in $V \times U = R^2$
  - $A = \{(x, y) \in R^2 | (x-1)^2 + (y-1)^2 \leq 1\}$
  - A_1 the projection of A on U: $[0, 2] \subset U$
  - A_2 the projection of A on V: $[0, 2] \subset V$

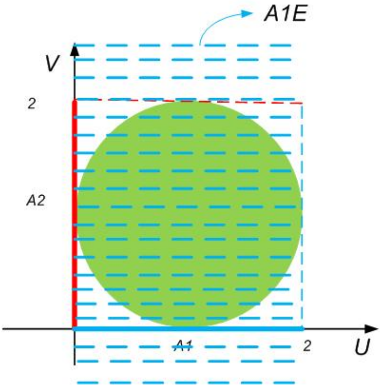


Projection of Fuzzy Sets

- Q: a fuzzy relation in $U_1 \times \ldots \times U_n$
- $\{i_1, \ldots, i_k\}$ a subsequence of $\{1, 2, \ldots, n\}$
- The projection of Q on $U_{i_1} \times \ldots \times U_{i_k}$ is a fuzzy relation Q_P in $U_{i_1} \times \ldots \times U_{i_k}$ s.t.:
  $\mu_{Q_p}(u_{i_1}, \ldots, u_{i_k}) = \underline{\max}_{u_{j_1} \in U_{j_1}, \ldots, u_{j(n-k)} \in U_{j(n-k)}} \mu_Q(u_1, \ldots, u_n)$
- $\{u_{j_1}, \ldots, u_{j(n-k)}\}$ is the complement of $\{u_{i_1}, \ldots, u_{i_k}\}$
- For binary fuzzy relation $U \times V$:
  - Q_1 projection in U: $\mu_{Q_1}(x) = \max_{y \in V} \mu_Q(x, y)$
- **Example:** Recall the "AE" example
  - Projection in U: $AE_1 = \int_U \max_{y \in V} e^{-(x-y)^2} / x = \int_U 1/x$
  - Projection in V: $AE_2 = \int_V \max_{x \in U} e^{-(x-y)^2} / y = \int_V 1/y$
- **Example:** Recall the "very far" example
  - Q_1: projection on u: $Q_1 = 0.9/Tehran + 0.95/Esfahan$
  - Q_2: projection on V:
    $Q_2 = 0.7/Tehran + 0.95/Tabriz + 0.8/Karaj + 0.5/Qom$

Cylindrical Extension

- Extending the projection of fuzzy cylindrically.
- **Example:** Recall the circle example
  - A_{1E} Cylindric extention of A_1 to $U \times V = R^2$
  - $A_{1E} = [0,1] \times (-\infty, \infty) \subset R^2$
- The projection constrains a fuzzy relation to a subspace
- The cylindric extension extends a fuzzy relation (or fuzzy set) from a subspace to the whole space.



- Let Q_p be a fuzzy relation in $U_{i_1} \times ... \times U_{i_k}$ and $\{i_1, \ldots, i_k\}$ is a subsequence of $\{1, 2, ..., n\}$, then the cylindric extension of Q_p to $U_l \times \ldots \times U_n$ is a fuzzy relation Q_{pE} in $U_l \times \ldots \times U_n$ $\mu_{Q_{pE}}(u_1, \ldots, u_n) = \mu_{Q_p}(u_{i_1}, \ldots, u_{i_k})$
- For binary set:
  - $U \times V$,
  - Q_1 a fuzzy set in U
  - Q_{1E} the cylindric extension to $U \times V$
  - $\mu_{Q_{1E}}(x, y) = \mu_{Q_1}(x)$
- **Example:** Recall the "AE" example
  - $A_{E_{1E}} = \int_{U\times V} 1/(x, y) = U \times V$
  - $A_{E_{2E}} = \int_{U\times V} 1/(x, y) = U \times V$

- **Example:** Recall the "very far" example
  - Q_{1E}: cylindrical ext. of Q_1 to $U \times V$:
    $Q_{1E} =$
    $0.9/(Tehran, Tehran) + 0.9(Tabriz, Tehran) + 0.9/(Karaj, Tehran) + 0.9/(Qom, Tehran) + 0.95/(Tehran, Esfahan), 0.95/(Tabriz, Esfahan) + 0.95/(Karaj, Esfahan) + 0.95/(Qom, Esfahan)$
  - Q_{1E}: cylindrical ext. of Q_2 to $U \times V$:
    $Q_{2E} =$
    $0.7/(Tehran, Tehran) + 0.7(Tehran, Esfahn) + 0.95/(Tabriz, Tehran) + 0.95/(Tabriz, Esfahan) + 0.8/(Karaj, Tehran), 0.8/(Karaj, Esfahan) + 0.5/(Qom, Tehran) + 0.5/(Qom, Esfahan)$

Cartesian Product of Fuzzy Sets

- Let $A_1, \ldots, A_n$ be fuzzy sets in $U_I, \ldots, U_n$, respectively. The Cartesian product of $A_1, \ldots, A_n$ denoted by $A_I \times \ldots \times A_n$, is a fuzzy relation in $U_I \times \ldots \times U_n$:
  $\mu_{A_1\times\ldots\times A_n}(u_1, \ldots, u_n) = \mu_{A_1}(u_1) * \ldots * \mu_{A_n}(u_n)$
- where $*$ represents any t-norm operator.
- **Lemma:** If Q is a fuzzy relation in $U_I \times \ldots \times U_n$ and $Q_I, \ldots, Q_n$ are its projections on $U_I, \ldots, U_n$, respectively, then
  $$Q \subset Q_1 \times \ldots \times Q_n$$

  where we use "min" for the t-norm

Cartesian Product of Fuzzy Sets

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  $\mu_{A_1 \times \ldots \times A_n}(u_1, \ldots, u_n) =$
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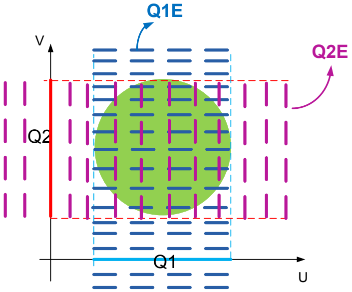
  where we use "min" for the t-norm

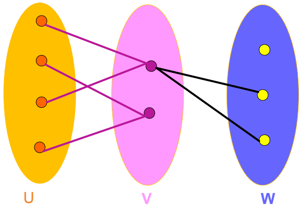


Composition

- $P(U,V)$ and $Q(V,W)$: two crisp binary relations that share a common set V.
- The composition of P and Q, (PoQ), is a relation in $U \times W$ s.t. $(x,z) \in Q$ iff there exists at least one $y \in V$ s.t. $(x,y) \in P$ and $(y,z) \in Q$.
- **Lemma**: PoQ is the composition of $P(U,V)$ and $Q(V,W)$ iff

$$\mu_{PoQ}(x,z) = \max t[\mu_P(x,y), \mu_Q(y,z)] \quad (1)$$

for any $(x,z) \in U \times W$, where t is any t-norm.



- Proof:
  - If PoQ is the composition:
    - $(x, z) \in PoQ \rightsquigarrow \exists y \in V$ s.t. $\mu_P(x, y) = 1 \& \mu_Q(y, z) = 1$
    - $\therefore \mu_{PoQ}(x, z) = 1 = \max_{y \in V} t[\mu_P(x, y), \mu_Q(y, z)]$
    - If $(x, z) \notin PoQ \rightsquigarrow$ for any $y \in V, \mu_P(x, y) = 0$ or $\mu_Q(y, z) = 0$
    - $\therefore \mu_{PoQ}(x, z) = 0 = max_{y \in V} t[\mu_P(x, y), \mu_Q(y, z)]$.
    - Eq. (1) is true.
  - Conversely, if the Eq. (1) is true:
    - $(x, z) \in PoQ \rightsquigarrow \max_{y \in V} t[\mu_P(x, y), \mu_Q(y, z)] = 1$
    - $\therefore$ there exists at least one $y \in V$ s.t. $\mu_P(x, y) = \mu_Q(Y, z) = 1$ (Axiom t1)
    - For $(x, z) \notin PoQ \rightsquigarrow \max_{y \in V}[\mu_P(x, y), \mu_Q(y, z)] = 0$
    - $\therefore \nexists y \in V$ s.t. $\mu_P(x, y) = \mu_Q(y, z) = 1$.
    - $\therefore PoQ$ is the composition

Fuzzy Composition

- Composition for fuzzy relations is defined similar to crisp relations
- Based on different definition of t-norm different composition is obtained.
- The two most popular compositions:
  - Max-Min: of fuzzy relations $P(U, V)$ and $Q(V, W)$ is a fuzzy relation PoQ in $U \times W$ s.t. $\mu_{PoQ}(x, z) = \max_{y \in V} \min[\mu_P(x, y), \mu_Q(y, z)]$
    - It uses the min for t-norm

    where $(x, z) \in U \times W$.
  - Max-Product: of fuzzy relations $P(U, V)$ and $Q(V, W)$ is a fuzzy relation PoQ in $U \times W$ s.t. $\mu_{PoQ}(x, z) = \max_{y \in V}[\mu_P(x, y).\mu_Q(y, z)]$ where $(x, z) \in U \times W$.
    - It uses algebraic product for t-norm

Example: Recall Dormitory example

- $V = \{Tehran, Tabriz, Karaj, Qom\}$, $U = \{Tehran, Esfahan\}$, $W = \{Boomehen, Kashan, Ardebil\}$
- $P(U, V)$ "very far"

$U \diagup V$	Tehran	Tabriz	Karaj	Qom
Tehran	0	0.9	0.1	0.3
Esfahan	0.7	0.95	0.8	0.5

Example: Recall Dormitory example

- $V = \{Tehran, Tabriz, Karaj, Qom\}$, $U = \{Tehran, Esfahan\}$, $W = \{Boomehen, Kashan, Ardebil\}$
- $P(U, V)$ "very far"

$U \diagup V$	Tehran	Tabriz	Karaj	Qom
Tehran	0	0.9	0.1	0.3
Esfahan	0.7	0.95	0.8	0.5

- $Q(V, W)$:"very near"

$V \diagup W$	Boomehen	Kashan	Ardebil
Tehran	1	0.4	0.1
Tabriz	0.2	0	0.8
Karaj	0.6	0.3	0.1
Qom	0.7	0.95	0

- $PoQ(U, W)$ using Max-min

P

U / V	Tehran	Tabriz	Karaj	Qom
Tehran	0	0.9	0.1	0.3
Esfahan	0.7	0.95	0.8	0.5

Q

$V \setminus W$	Boomehen	Kashan	Ardebil
Tehran	1	0.4	0.1
Tabriz	0.2	0	0.8
Karaj	0.6	0.3	0.1
Qom	0.7	0.95	0

Min

0	**0.9**	**0.1**	**0.3**
1	**0.2**	**0.6**	**0.7**
0	**0.2**	**0.1**	**0.3**

Max

PoQ

U / W	Boomehen	Kashan	Ardebil
Tehran	0.3	0.3	0.8
Esfahan	0.7	0.5	0.8

- $PoQ(U, W)$ using Max-Product

P

U/V	Tehran	Tabriz	Karaj	Qom
Tehran	0	0.9	0.1	0.3
Esfahan	0.7	0.95	0.8	0.5

Q

$V \setminus W$	Boomehen	Kashan	Ardebil
Tehran	1	0.4	0.1
Tabriz	0.2	0	0.8
Karaj	0.6	0.3	0.1
Qom	0.7	0.95	0

	0	**0.9**	**0.1**	**0.3**
x	**1**	**0.2**	**0.6**	**0.7**
	0	**0.18**	**0.06**	**0.21**

Max

PoQ

U/W	Boomehen	Kashan	Ardebil
Tehran	0.21	0.0.285	0.72
Esfahan	0.7	0.475	0.76

- The relational matrix for the fuzzy composition PoQ can be computed according to the following method:
  - For max-min composition
    - write out each element in the matrix product PQ, But treat:
    - each multiplication as a min operation
    - each addition as a max operation
  - For max-product composition,
    - write out each element in the matrix product PQ, but treat
    - each addition as a max operation.

Extension Principle

- Objective: the domain of a function be extended from crisp points in U to fuzzy sets in U
- $f : U \rightarrow V$ a function from crisp set U to crisp set V.
- A: a fuzzy set U
- $B = f(A)$ a fuzzy set in V
  - If f is an one-to-one mapping $\mu_B(y) = \mu_A[f^{-1}(y)], y \in V$
  - where $f[f^{-1}(y)] = y$
- If f is not one-by-one what should we do ?:(
- Example:$f(x_1) = f(x_2) = y,\ x_1 \neq x_2 \rightsquigarrow \mu_A(x_1) \neq \mu_A(x_2)$
  - Two different values is obtained for $\mu_B(y)$

Extension Principle

- Extension Principle: $\mu_B(y) = \max_{x \in f^{-1}(y)} \mu_A(x), y \in V$
  - $f^{-1}(y)$: set of all points $x \in U$ s.t. $f(x) = y$
- **Example** $U = \{1, \ldots, 10\}, x \in U, f(x) = x^2 \in V = \{1, \ldots, 100\}$
- Fuzzy set: " small"$= 1/1 + 1/2 + 0.8/3 + 0.6/4 + 0.4/5$
- $\therefore$ "small$^2 = 1/1 + 1/4 + 0.8/9 + 0.6/16 + 0.4/25$