Nonlinear Control
Lecture 9: Feedback Linearization

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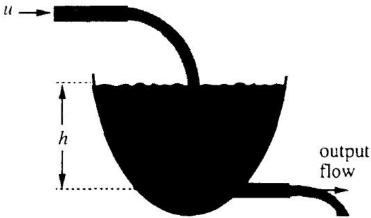
Feedback Linearzation

- **The main idea is:** algebraically transform a nonlinear system dynamics into a (fully or partly) linear one, so that linear control techniques can be applied.
- In its simplest form, feedback linearization cancels the nonlinearities in a nonlinear system so that the closed-loop dynamics is in a linear form.
- **Example:** Controlling the fluid level in a tank
  - Objective: controlling of the level h of fluid in a tank to a specified level h_d, using control input u
  - the initial level is h_0.



Fluid level control in a tank

Example Cont'd

- The dynamics:

$$A(h)\dot{h}(t) = u - a\sqrt{2gh}$$

where $A(h)$ is the cross section of the tank and a is the cross section of the outlet pipe.

- Choose $u = a\sqrt{2gh} + A(h)v \rightsquigarrow \dot{h} = v$
- Choose the equivalent input v: $v = -\alpha\tilde{h}$ where $\tilde{h} = h(t) - h_d$ is error level, α a pos. const.
- $\therefore$ resulting closed-loop dynamics: $\dot{h} + \alpha\tilde{h} = 0 \Rightarrow \tilde{h} \to 0$ as $t \to \infty$
- The actual input flow: $u = a\sqrt{2gh} + A(h)\alpha\tilde{h}$
  - First term provides output flow $a\sqrt{2gh}$
  - Second term raises the fluid level according to the desired linear dynamics
- If h_d is time-varying: $v = \dot{h}_d(t) - \alpha\tilde{h}$
  - $\therefore$ $\tilde{h} \to 0$ as $t \to \infty$

- Canceling the nonlinearities and imposing a desired linear dynamics, can be simply applied to a class of nonlinear systems, so-called companion form, or controllability canonical form:
- A system in companion form:

$$x^{(n)}(t) = f(\mathbf{x}) - b(\mathbf{x})u \quad (1)$$

  - u is the scalar control input
  - x is the scalar output;$\mathbf{x} = [x, \dot{x}, ..., x^{(n-1)}]$ is the state vector.
  - $f(x)$ and $b(x)$ are nonlinear functions of the states.
  - no derivative of input u presents.
- (1) can be presented as controllability canonical form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ f(x) + b(x)u \end{bmatrix}$$

- for nonzero b, define control input: $u = \frac{1}{b}[v - f]$

Feedback Linearzation

- $\therefore$ the control law:

$$v = -k_0 x - k_1 \dot{x} - \ldots - k_{n-1} x^{(n-1)}$$

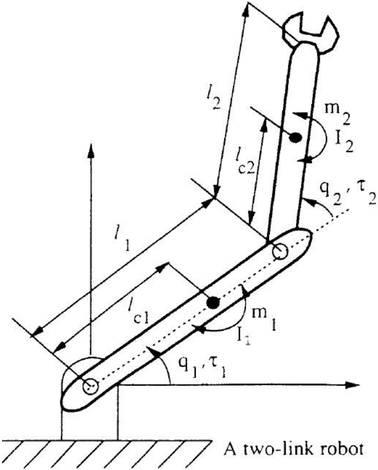
  - k_i is chosen s.t. the roots of $s^n + k_{n-1}s^{n-1} + \ldots + k_0$ are strictly in LHP.
- **Thus**: $x^{(n)} + k_{n-1}x^{(n-1)} + \ldots + k_0 = 0$ is e.s.
- For tracking desired output x_d, the control law is:

$$v = x_d^{(n)} - k_0 x - k_1 \dot{x} - \ldots - k_{n-1} x^{(n-1)}$$

- $\therefore$ Exponentially convergent tracking, $e = x - x_d \to 0$.
- This method is extendable when the scalar x was replaced by a vector and the scalar b by an invertible square matrix.
- When u is replaced by an invertible function $g(u) \rightsquigarrow u = g^{-1}(\frac{1}{b}[v - f])$,

Example: Feedback Linearization of a Two-link Robot

- A two-link robot: each joint equipped with
  - a motor for providing input torque
  - an encoder for measuring joint position
  - a tachometer for measuring joint velocity
- objective: the joint positions q_l and q_2 follow desired position histories $q_{dl}(t)$ and $q_{d2}(t)$
- For example when a robot manipulator is required to move along a specified path, e.g., to draw circles.



A two-link robot

- Using the Lagrangian equations the robotic dynamics are:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h\dot{q}_2 - h\dot{q}_1 \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

where $q = [q_1 \quad q_2]^T$: the two joint angles, $\tau = [\tau_1 \quad \tau_2]^T$: the joint inputs, and

$$\begin{aligned} H_{11} &= m_1 l_{c1}^2 + I_1 + m_2[l_1^2 + l_{c2}^2 + 2l_1 l_{c2} \cos q_2] + I_2 \\ H_{22} &= m_2 l_{c2}^2 + I_2 H_{12} = H_{21} = m_2 l_1 l_{c_2} \cos q_2 + m_2 l_{c2}^2 + I_2 \\ g_1 &= m_1 l_{c1} \cos q_1 + m_2 g[l_{c2} \cos(q_1 + q_2) + l_1 \cos q_1] \\ g_2 &= m_2 l_{c2} g \cos(q_1 + q_2), \;\; h = m_2 l_1 l_{c2} \sin q_2 \end{aligned}$$

- Control law for tracking, (computed torque):

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h\dot{q}_2 - h\dot{q}_1 \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

where $v = \ddot{q}_d - 2\lambda\dot{\tilde{q}} - \lambda^2\tilde{q}$, $\tilde{q} = q - q_d$: position tracking error, λ: pos. const.

- $\therefore$ $\ddot{\tilde{q}}_d + 2\lambda\dot{\tilde{q}} + \lambda^2\tilde{q} = 0$ where $\tilde{q}$ converge to zero exponentially.
- This method is applicable for arbitrary # of links

Input-State Linearization

- When the nonlinear dynamics is not in a controllability canonical form, use algebraic transformations
- Consider the SISO system

$$\dot{x} = f(x, u)$$

- In input-state linearization technique:
  1. finds a state transformation $z = z(x)$ and an input transformation $u = u(x, v)$ s.t. the nonlinear system dynamics is transformed into $\dot{z} = Az + bv$
  2. Use standard linear techniques (such as pole placement) to design v.

Example:

- Consider

$$\begin{aligned} \dot{x}_1 &= -2x_1 + ax_2 + \sin x_1 \\ \dot{x}_2 &= -x_2 \cos x_1 + u \cos(2x_1) \end{aligned}$$

  - Equ. pt. $(0, 0)$
  - The nonlinearity cannot be directly canceled by the control input u
  - Define a new set of variables:

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= ax_2 + \sin x_1 \\ \therefore \quad \dot{z}_1 &= -2z_1 + z_2 \\ \dot{z}_2 &= -2z_1 \cos z_1 + \cos z_1 \sin z_1 + au \cos(2z_1) \end{aligned}$$

  - The Equ. pt. is still $(0, 0)$.
  - The control law: $u = \frac{1}{a\cos(2z_1)}(v - \cos z_1 \sin z_1 + 2z_1 \cos z_1)$
  - The new dynamics is linear and controllable: $\dot{z}_1 = -2z_1 + z_2, \quad \dot{z}_2 = v$
  - By proper choice of feedback gains k_1 and k_2 in $v = -k_1 z_1 - k_2 z_2$, place the poles properly.
  - Both z_1 and z_2 converge to zero, $\rightsquigarrow$ the original state x converges to zero

- The result is not global.
  - The result is not valid when $x_l = (\pi/4 \pm k\pi/2),\ k = 0, 1, 2, ...$
- The input-state linearization is achieved by a combination of a state transformation and an input transformation with state feedback used in both.
- To implement the control law, the new states (z_1, z_2) must be available.
  - If they are not physically meaningful or measurable, they should be computed by measurable original state x.
- If there is uncertainty in the model, e.g., on $a \rightsquigarrow$ error in the computation of new state z as well as control input u.
- For tracking control, the desired motion needs to be expressed in terms of the new state vector.
- Two questions arise for more generalizations which will be answered in next lectures:
  - What classes of nonlinear systems can be transformed into linear systems?
  - How to find the proper transformations for those which can?

Input-Output Linearization

- Consider

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned}$$

- Objective: tracking a desired trajectory $y_d(t)$, while keeping the whole state bounded
- $y_d(t)$ and its time derivatives up to a sufficiently high order are known and bounded.
- The difficulty: output y is only *indirectly* related to the input u
  - $\therefore$ it is not easy to see how the input u can be designed to control the tracking behavior of the output y.
- Input-output linearization approach:
  1. Generating a linear input-output relation
  2. Formulating a controller based on linear control

Example:

- Consider

$$\begin{aligned}\dot{x}_1 &= \sin x_2 + (x_2+1)x_3 \\ \dot{x}_2 &= x_1^5 + x_3 \\ \dot{x}_3 &= x_1^2 + u \\ y &= x_1\end{aligned}$$

- To generate a direct relationship between the output y and the input u, differentiate the output $\dot{y} = \dot{x}_1 = \sin x_2 + (x_2+1)x_3$
- No direct relationship $\rightsquigarrow$ differentiate again: $\ddot{y} = (x_2+1)u + f(x)$, where $f(x) = (x_1^5 + x_3)(x_3 + \cos x_2) + (x_2+1)x_1^2$
- Control input law: $u = \frac{1}{x_2+1}(v - f)$.
- Choose $v = \ddot{y}_d - k_1 e - k_2 \dot{e}$, where $e = y - y_d$ is tracking error, k_1 and k_2 are pos. const.
- The closed-loop system: $\ddot{e} + k_2\dot{e} + k_1 e = 0$
- $\therefore$ e.s. of tracking error

Example Cont'd

- The control law is defined everywhere except at singularity points s.t. $x_2 = -1$
- To implement the control law, full state measurement is necessary, because the computations of both the derivative y and the input transformation need the value of x.
- If the output of a system should be differentiated r times to generate an explicit relation between y and u, the system is said to have relative degree r.
  - For linear systems this terminology expressed as $\#$ poles $-\#$ zeros.
- For any controllable system of order n, by taking at most n differentiations, the control input will appear to any output, i.e., $r \leq n$.
  - If the control input never appears after more than n differentiations, the system would not be controllable.

Feedback Linearzation

- **Internal dynamics**: a part of dynamics which is unobservable in the input-output linearization.
  - In the example it **can be** $\dot{x}_3 = x_1^2 + \frac{1}{x_2+1}(\ddot{y}_d(t) - k_1 e - k_2 \dot{e} + f)$
- The desired performance of the control based on the reduced-order model depends on the stability of the internal dynamics.
  - stability in BIBO sense
- **Example:** Consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 + u \\ u \end{bmatrix} \qquad (2)$$

$$y = x_1$$

- Control objective: y tracks y_d.
  - First differentiations of $y \rightsquigarrow$ linear I-O relation
  - The control law $u = -x_2^3 - e(t) - \dot{y}_d(t) \rightsquigarrow$ exp. convergence of e : $\dot{e} + e = 0$
  - Internal dynamics: $\dot{x}_2 + x_2^3 = \dot{y}_d - e$
  - Since e and $\dot{y}_d$ are bounded $(\dot{y}_d(t) - e \leq D)$ x_2 is ultimately bounded.

- I-O linearization can also be applied to stabilization ($y_d(t) \equiv 0$):
  - For previous example the objective will be y and $\dot{y}$ will be driven to zero and stable internal dynamics guarantee stability of the whole system.
  - No restriction to choose physically meaningful $h(x)$ in $y = h(x)$
  - Different choices of output function leads to different internal dynamics which some of them may be unstable.
- When the relative degree of a system is the same as its order:
  - There is no internal dynamics
  - The problem will be input-state linearization

Summary

- Feedback linearization cancels the nonlinearities in a nonlinear system s.t. the closed-loop dynamics is in a linear form.
- Canceling the nonlinearities and imposing a desired linear dynamics, can be applied to a class of nonlinear systems, named companion form, or controllability canonical form.
- When the nonlinear dynamics is not in a controllability canonical form, input-state linearization technique is employed:
  1. Transform input and state into companion canonical form
  2. Use standard linear techniques to design controller
- For tracking a desired traj, when y is not directly related to u, I-O linearizaton is applied:
  1. Generating a linear input-output relation (take derivative of y $r \leq n$ times)
  2. Formulating a controller based on linear control
- Relative degree: $\#$ of differentiating y to find explicate relation to u.
- If $r \neq n$, there are $n - r$ internal dynamics that their stability be checked.

Internal Dynamics of Linear Systems

- In general, directly determining the stability of the internal dynamics is not easy since it is nonlinear. nonautonomous, and coupled to the "external" closed-loop dynamics.
- We are seeking to translate the concept of of internal dynamics to the more familiar context of linear systems.
- **Example:** Consider the controllable, observable system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} x_2 + u \\ u \end{bmatrix} \\ y &= x_1 \end{aligned} \tag{3}$$

- Control objective: y tracks y_d.
  - First differentiations of $y \rightsquigarrow \dot{y} = x_2 + u$
  - The control law $u = -x_2 - e(t) - \dot{y}_d(t) \rightsquigarrow$ exp. convergence of $e : \dot{e} + e = 0$
  - Internal dynamics: $\dot{x}_2 + x_2 = \dot{y}_d - e$
  - e and $\dot{y}_d$ are bounded $\rightsquigarrow$ x_2 and therefore u are bounded.

- Now consider a little different dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + u \\ -u \end{bmatrix} \qquad (4)$$
$$y = x_1$$

- using the same control law yields the following internal dynamics

$$\dot{x}_2 - x_2 = e(t) - \dot{y}_d$$

- Although y_d and y are bounded, x_2 and u diverge to ∞ as $t \to \infty$
- why the same tracking design method yields different results?
  - Transfer function of (3) is: $W_1(s) = \frac{s+1}{s^2}$.
  - Transfer function of (4) is: $W_2(s) = \frac{s-1}{s^2}$.
  - $\therefore$ Both have the same poles but different zeros
  - The system W_1 which is minimum-phase tracks the desired trajectory perfectly.
  - The system W_2 which is nonminimum-phase requires infinite effort for tracking.

Internal Dynamics

- Consider a third-order linear system with one zero

$$\dot{x} = Ax + bu, \quad y = c^T x \tag{5}$$

- Its transfer function is: $y = \frac{b_0 + b_1 s}{a_0 + a_1 s + a_2 s^2 + a_3 s^3} u$
- First transform it into the companion form:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{6}$$

$$y = [b_0 \ b_1 \ 0] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

- In second derivation of y, u appears:
  $\ddot{y} = b_0 z_3 + b_1(-a_0 z_1 - a_1 z_2 - a_2 z_3 + u)$
- $\therefore$ Required number of differentiations (the relative degree) is indeed the same as # of poles- # of zeros
  - Note that: the I-O relation is independent of the choice of state variables $\rightsquigarrow$ two differentiations is required for u to appear if we use (5).
- The control law: $u = (a_0 z_1 + a_1 z_2 + a_2 z_3 - \frac{b_0}{b_1} z_3) + \frac{1}{b_1}(-k_1 e - k_2 \dot{e} - \ddot{y}_d)$
- $\therefore$ an exp. stable tracking is guaranteed
- The internal dynamics can be described by only one state equation
  - z_1 can complete the state vector,(z_1, y, and $\dot{y}$ are related to z_1, z_2 and z_3 through a one-to-one transformation).
  - $\dot{z}_1 = z_2 = \frac{1}{b_1}(y - b_0 z_1)$
  - y is bounded $\rightsquigarrow$ stability of the internal dynamics depends on $-\frac{b_0}{b_1}$
  - If the system is minimum phase the internal dynamics is stable (independent of initial conditions and magnitude of desired trajectory)

Zero-Dynamics

- For linear systems the stability of the internal dynamics is determined by the locations of the zeros.
- To extend the results for nonlinear systems the concept of zero should be modified.
- Extending the notion of zeros to nonlinear systems is not trivial
  - In linear systems I-O relation is described by transfer function which zeros and poles are its fundamental components. **But** in nonlinear systems we cannot define transfer function
  - Zeros are intrinsic properties of a linear plant. **But** for nonlinear systems the stability of the internal dynamics may depend on the specific control input.
- Zero dynamics: is defined to be the internal dynamics of the system when the system output is **kept** at zero by the input.(output and all of its derivatives)

- For dynamics (2), the zero dynamics is $\dot{x}_2 + x_2^3 = 0$
  - we find input u to maintain the system output at *zero uniquely* (keep x_1 zero in this example),
  - By Layap. Fcn $V = x_2^2$ it can be shown it is a.s
- For linear system (5), the zero dynamics is $\dot{z}_1 + (b_0/b_1)z_1 = 0$
- $\therefore$ The poles of the zero-dynamics are exactly the zeros of the system.
- In linear systems, if all zeros are in LHP $\rightsquigarrow$ g.a.s. of the zero-dynamics $\rightsquigarrow$ g.s. of internal dynamics.
- In nonlinear systems, no results on the global stability
  - only local stability is guaranteed for the internal dynamics even if the zero-dynamics is g.e.s.
- Zero-dynamics is an intrinsic feature of a nonlinear system, which does not depend on the choice of control law or the desired trajectories.
- Examining the stability of zero-dynamics is easier than examining the stability of internal dynamics,**But** the result is local.
  - Zero-dynamics only involves the internal states
  - Internal dynamics is coupled to the external dynamics and desired trajs.

Zero-Dynamics

- Similar to the linear case, a nonlinear system whose zero dynamics is asymptotically stable is called an asymptotically minimum phase system,
- If the zero-dyiamics is unstable, different control strategies should be sought
- As summary control design based on input-output linearization is in three steps:
  1. Differentiate the output y until the input u appears
  2. Choose u to cancel the nonlinearities and guarantee tracking convergence
  3. Study the stability of the internal dynamics
- If the relative degree associated with the input-output linearization is the same as the order of the system $\rightsquigarrow$ the nonlinear system is fully linearized $\rightsquigarrow$ satisfactory controller
- Otherwise, the nonlinear system is only partly linearized $\rightsquigarrow$ whether or not the controller is applicable depends on the stability of the internal dynamics.

Preliminary Mathematics

- To formalize and generalize the previous intuitive concepts for a broad class of nonlinear systems, let us introduce some mathematical tools.
- Vector function $\mathbf{f} : R^n \rightarrow R^n$ is called a vector field in R^n.
- Smooth vector field: function $\mathbf{f}(x)$ has continuous partial derivatives of any required order.
- Gradient of a smooth scalar function $h(x)$ is denoted by

$$\nabla h = \frac{\partial h}{\partial \mathbf{x}}, \quad \text{where } (\nabla h)_j = \frac{\partial h}{\partial x_j}$$

- Jacobian of a vector field $\mathbf{f}(\mathbf{x})$ is $\nabla \mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$, where $(\nabla \mathbf{f})_j = \frac{\partial f_i}{\partial x_j}$
- **Lie derivative of h with respect to f** is a scalar function defined by $L_{\mathbf{f}} h = \nabla h \mathbf{f}$, where $h : R^n \rightarrow R$ is a smooth scalar and $\mathbf{f} : R^n \rightarrow R^n$ is a smooth vector field.
- If $\mathbf{g}$ is another vector field: $L_{\mathbf{g}} L_{\mathbf{f}} h = \nabla (L_{\mathbf{f}} h) \mathbf{g}$

- **Example:** For single output system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad y = h(\mathbf{x})$ then

$$\dot{y} = \frac{\partial h}{\partial \mathbf{x}}\dot{\mathbf{x}} = L_{\mathbf{f}} h$$
$$\ddot{y} = \frac{\partial [L_{\mathbf{f}} h]}{\partial \mathbf{x}}\dot{\mathbf{x}} = L_{\mathbf{f}}^2 h$$

- If V is a Lyap. fcn candidate, its derivative $\dot{V}$ can be written as $L_{\mathbf{f}} V$.
- **Lie bracket of f and g** is a third vector field defined by $[\mathbf{f}, \ \mathbf{g}] = \nabla \mathbf{g} \mathbf{f} - \nabla \mathbf{f}\ \mathbf{g}$, where **f** and **g** two vector field on R^n.
- The Lie bracket $[\mathbf{f}, \ \mathbf{g}]$ is also written as $ad_{\mathbf{f}}\ \mathbf{g}$ (where ad stands for "adjoint").
- **Example:** Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u$ where

$$\mathbf{f} = \begin{bmatrix} -2x_1 + ax_2 + \sin x_1 \\ -x_2 \cos x_1 \end{bmatrix}, \mathbf{g} = \begin{bmatrix} 0 \\ \cos(2x_1) \end{bmatrix}$$

- So the Lie bracket is:

$$[\mathbf{f}, \ \mathbf{g}] = \begin{bmatrix} -a\cos(2x_1) \\ \cos x_1 cos(2x_1) - 2\sin(2x_1)(-2x_1 + ax_2 + \sin x_1) \end{bmatrix}$$

- **Lemma:** *Lie brackets have the following properties:*
  1. *bilinearity:*

$$[\alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2, \mathbf{g}] = \alpha_1 [\mathbf{f}_1, \mathbf{g}] + \alpha_2 [\mathbf{f}_2, \mathbf{g}]$$

$$[\mathbf{f}, \alpha_1 \mathbf{g}_1 + \alpha_2 \mathbf{g}_2] = \alpha_1 [\mathbf{f}, \mathbf{g}_1] + \alpha_2 [\mathbf{f}, \mathbf{g}_2]$$

  where $\mathbf{f}$, $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{g}$ $\mathbf{g}_1$, $\mathbf{g}_2$ *are smooth vector fields and* α_1 *and* α_2 *are constant scalars.*

  2. *skew-commutativity:*

$$[\mathbf{f}, \mathbf{g}] = -[\mathbf{g}, \mathbf{f}]$$

  3. *Jacobi identity*

$$L_{ad_{\mathbf{f}}\mathbf{g}} h = L_{\mathbf{f}} L_{\mathbf{g}} h - L_{\mathbf{g}} L_{\mathbf{f}} h$$

  where h is a smooth fcn.

Diffeomorphism

- The concept of diffeomorphism can be applied to transform a nonlinear system into another nonlinear system in terms of a new set of states.
- **Definition:** *A function $\phi : \mathcal{R}^n \rightarrow \mathcal{R}^n$ defined in a region Ω is called a diffeomorphism if it is smooth, and if its inverse ϕ^{-1} exists and is smooth.*
- If the region Ω is the whole space $\mathcal{R}^n \rightsquigarrow \phi(x)$ is *global diffeomorphism*
- Global diffeomorphisms are rare,we are looking for *local diffeomorphisms*.
- **Lemma:** *Let $\phi(x)$ be a smooth function defined in a region Ω in $\mathcal{R}^n$. If the Jacobian matrix $\nabla\phi$ is non-singular at a point $x = x_0$ of Ω, then $\phi(x)$ defines a local diffeomorphism in a subregion of Ω*

Diffeomorphism

- Consider the dynamic system described by

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

- Let the new set of states $z = \phi(x) \rightsquigarrow \dot{z} = \frac{\partial \phi}{\partial x}\dot{x} = \frac{\partial \phi}{\partial x}(f(x) + g(x)u)$
- The new state-space representation

$$\dot{z} = f^*(z) + g^*(z)u, \quad y = h^*(z)$$

  where $x = \phi^{-1}(z)$.
- **Example of a non-global diffeomorphism**: Consider

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \phi(x) = \begin{bmatrix} 2x_1 + 5x_1x_2^2 \\ 3\sin x_2 \end{bmatrix}$$

  - Its Jacobian matrix: $\frac{\partial \phi}{\partial x} = \begin{bmatrix} 2 + 5x_2^2 & 10x_1x_2 \\ 0 & 3\cos x_2 \end{bmatrix}$.
  - rank is 2 at $x = (0, 0) \rightsquigarrow$ local diffeomorphism around the origin where $\Omega = \{(x_1, x_2), |x_2| < \pi/2\}$.
  - outside the region, the inverse of ϕ does not uniquely exist.

Frobenius Theorem

- An important tool in feedback linearization
- Provide necess. and suff. conditions for solvability of PDEs.
- Consider a PDE with (n=3):

$$\begin{aligned} \frac{\partial h}{\partial x_1} f_1 + \frac{\partial h}{\partial x_2} f_2 + \frac{\partial h}{\partial x_3} f_3 &= 0 \\ \frac{\partial h}{\partial x_1} g_1 + \frac{\partial h}{\partial x_2} g_2 + \frac{\partial h}{\partial x_3} g_3 &= 0 \end{aligned} \tag{7}$$

  where $f_i(x_1, x_2, x_3)$, $g_i(x_1, x_2, x_3)$ $(i = 1,\ 2,\ 3)$ are known scalar fcns and $h(x_1, x_2, x_3)$ is an unknown function.
- This set of PDEs is uniquely determined by the two vectors $f = [f_1\ f_2\ f_3]^T$, $g = [g_1\ g_2\ g_3]^T$.
- If the solution $h(x_1, x_2, x_3)$ exists, the set of vector fields $\{f,\ g\}$ is completely integrable.
- When the equations are solvable?

Frobenius Theorem

- Frobenius theorem states that Equation (7) has a solution $h(x_1, x_2, x_3)$ iff there exists **scalar functions** $\alpha_1(x_1, x_2, x_3)$ and $\alpha_2(x_1, x_2, x_3)$ such that
$$[f,\ g] = \alpha_1\ f + \alpha_2\ g$$
i.e., if the Lie bracket of f and g can be expressed as a linear combination of **f** and **g**
- This condition is called *involutivity of the vector fields* $\{f,\ g\}$.
- Geometrically, it means that the vector field $[f,\ g]$ is in the plane formed by the two vectors **f** and **g**
- The set of vector fields $\{f,\ g\}$ is completely integrable iff it is involutive.
- **Definition (Complete Integrability):** *A linearly independent set of vector fields* $\{f_1,\ f_2, ...,\ f_m\}$ *on* R^n *is said to be completely integrable, iff, there exist* $n - m$ *scalar fcns* $h_1(x),\ h_2(x), ...,\ h_{n-m}(x)$ *satisfying the system of PDEs:*
$$\nabla h_i\ f_j\ = 0$$
where $1\ \leq\ i\ \leq\ n - m,\ 1\ \leq\ j\ \leq\ m$ *and* ∇h_i *are linearly independent.*

- Number of vectors: **m**, dimension of the vectors: **n**, number of unknown scalar fcns h_i: **(n-m)**, number of PDEs: **m(n-m)**
- **Definition (Involutivity):** *A linearly independent set of vector fields* $\{f_1,\ f_2, \ldots,\ f_m\}$ *on* R^n *is said to be involutive iff, there exist scalar fcns* $\alpha_{ijk}\ :\ R^N \longrightarrow R$ *s.t.*

$$[f_i,\ f_j](x) = \sum_{k=i}^{m} \alpha_{ijk}(x)\ f_k(x) \qquad \forall\ i,j$$

i.e., the Lie bracket of any two vector fields from the set $\{f_1,\ f_2, \ldots,\ f_m\}$ *can be expressed as the linear combination of the vectors from the set.*

  - Constant vector fields are involutive since their Lie brackets are zero
  - A set composed of a single vector is involutive:

$$[f,\ f] = (\nabla f)f - (\nabla f)f = 0$$

  - Involutivity means:

$$rank\,(f_1(x)\ \ldots.\ f_m(x)) = rank\,(f_1(x)\ \ldots.\ f_m(x)\ [f_i,\ f_j](x))$$

for all x and for all $i,\ j$.

Frobenius Theorem

- **Theorem (Frobenius):** *Let $f_1,\ f_2,\ ....,\ f_m$ be a set of linearly independent vector fields. The set is completely integrable* **iff** *it is involutive.*
- **Example:** Consider the set of PDEs:

$$
\begin{aligned}
4x_3\frac{\partial h}{\partial x_1} - \frac{\partial h}{\partial x_2} &= 0 \\
-3x_1\frac{\partial h}{\partial x_1} + (-4x_3^2 - 3x_2)\frac{\partial h}{\partial x_2} + 2x_3\frac{\partial h}{\partial x_3} &= 0
\end{aligned}
$$

- The associated vector fields are $\{f_1,\ f_2\}$

$$f_1 = [4x_3\ \ -1\ \ 0]^T \qquad f_2 = [-3x_1\ (-4x_3^2 - 3x_2)\ \ 2x_3]^T$$

- We have $[f_1,\ f_2] = [-12x_3\ 3\ 0]^T$
- Since $[f_1,\ f_2] = -3f_1 + 0f_2$, the set $\{f_1,\ f_2\}$ is involutive and the set of PDEs are solvable.

Input-State Linearization

- Consider the following SISO nonlinear system
$$\dot{x} = f(x) + g(x)u \tag{8}$$
where f and g are smooth vector fields
- The above system is also called "linear in control" or "affine"
- If we deal with the following class of systems:
$$\dot{x} = f(x) + g(x)w(u + \phi(x))$$
where w is an invertible scalar fcn and ϕ is an arbitrary fcn
  - We can use $v = w(u + \phi(x))$ to get the form (8).
  - Control design is based on v and u can be obtained by inverting w:
$$u = w^{-1}(v) - \phi(x)$$
- Now we are looking for
  - Conditions for system linearizability by an input-state transformation
  - A technique to find such transformations
  - A method to design a controller based on such linearization technique

Input-State Linearization

- **Definition: Input-State Linearization** *The nonlinear system (8) where $f(x)$ and $g(x)$ are smooth vector fields in R^n is input-state linearizable if there exist region Ω in R^n, a diffeomorphism mapping $\phi : \Omega \longrightarrow R^n$, and a control law:*

$$u = \alpha(x) + \beta(x)v$$

s.t. new state variable $z = \phi(x)$ and new input variable v satisfy an LTI relation:

$$\begin{aligned} \dot{z} &= Az + Bv \\ A &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \end{aligned} \tag{9}$$

- The new state z is called the *linearizing state* and the control law u is called the *linearizing control law*
- Let $z = z(x)$

- (9) is the so-called controllability or companion form
- The companion form can be obtained from any form by a transformation $\Longrightarrow$ the above form is a general form
- This form is an special case of Input-Output linearization leading to relative degree $r = n$.
- Hence, if the system I-O linearizable with $r = n$, it is also I-S linearizable.
- On the other hand, if the system is I-S linearizable, it is also I-O linearizable with $y = z, \; r = n$.
- **Lemma:** *An n^{th} order nonlinear system is I-S linearizable* **iff** *there exists a scalar fcn $z_1(x)$ for which the system is I-O linearizable with $r = n$.*
- **Conditions for Input-State Linearization:**
  - **Theorem:** *The nonlinear system (8) with $f(x)$ and $g(x)$ being smooth vector field is input-state linearizable* **iff** *there exists a region Ω s.t. the following conditions hold:*
  - *The vector fields $\{g, \; ad_f g, \; \dots \; ad_f{}^{n-1} g\}$ are linearly independent in Ω*
  - *The set $\{g, \; ad_f g, \; \dots \; ad_f{}^{n-2} g\}$ is involutive in Ω*

- The first condition can be interpreted as a controllability condition
- For linear system, the vector field above becomes $\{B,\ AB,\ \dots\ A^{n-1}B\}$
- Linear independency is equivalent to invertibility of controllability matrix
- The second condition is always satisfied for linear systems since the vector fields are constant, but for nonlinear system is not necessarily satisfied.
- It is necessary according to Ferobenius theorem for existence of $z_1(x)$.
- **Lemma:** *If $z(x)$ is a smooth vector field in Ω, then the set of equations*

$$L_g z = L_g L_f z = \ \dots\ L_g L_f{}^k z = 0$$

is equivalent to

$$L_g z = L_{ad_f\ g} z = \ \dots\ L_{ad_f\ {}^k g} z = 0$$

- **Proof:**
  - Let $k = 1$, from Jacobi's identity, we have

$$L_{ad_f\ g} z = L_f L_g z - L_g L_f z = 0 - 0 = 0$$

- When $k = 2$, we have from Jacobi's identity:

$$L_{ad_f{}^2 g} z = L_f{}^2 L_g z - 2 L_f L_g L_f z + L_g L_f{}^2 z = 0 - 0 + 0 = 0$$

- **Proof of the linearization theorem:**
- **Necessity:**
  - Suppose state transformation $z = z(x)$ and input transformation $u = \alpha(x) + \beta(x) v$ s.t. z and v satisfy (9), i.e.

$$\dot{z}_1 = \frac{\partial z_1}{\partial x}(f + gu) = z_2$$

similarly:

$$\begin{aligned}
\frac{\partial z_1}{\partial x} f + \frac{\partial z_1}{\partial x} gu &= z_2 \\
\frac{\partial z_2}{\partial x} f + \frac{\partial z_2}{\partial x} gu &= z_3 \\
&\vdots \\
\frac{\partial z_n}{\partial x} f + \frac{\partial z_n}{\partial x} gu &= v
\end{aligned}$$

- $z_1, \ \dots, \ z_n$ are independent of u, but v is not, hence:

$$\begin{aligned} L_g z_1 &= L_g z_2 = \dots L_g z_{n-1} = 0, \quad L_g z_n \neq 0 \\ L_f z_i &= z_{i+1}, \quad i = 1, 2, \dots, n-1 \end{aligned}$$

- Use, $z = [z_1 \ L_f z_1, \ \dots \ L_f{}^{n-1} z_1]^T$ to get

$$\begin{aligned} \dot{z}_k &= z_{k+1}, \quad k = 1, \ \dots \ n-1 \\ \dot{z}_n &= L_f{}^n z_1 + L_g L_f{}^{n-1} z_1 u \end{aligned}$$

- The above equations can be expressed in terms of z_1 only

$$\nabla z_1 ad_f{}^k g = 0, \quad k = 0, \ 1, \ 2, \ \dots, \ n-2 \tag{10}$$

$$\nabla z_1 ad_f{}^{n-1} g = (-1)^{n-1} L_g L_f \tag{11}$$

- First note that for above eqs to hold, the vector field $g, \ ad_f \ g, \ \dots, \ ad_f{}^{n-1} g$ must be linearly independent.
- If for some $i (i \leq n-1)$ there exist scalar fcns $\alpha_1(x), \ \dots \ \alpha_{i-1}(x)$ s.t.

$$ad_f{}^i g = \sum_{k=0}^{i-1} \alpha_k ad_f{}^k g$$

- We, then have:

$$\therefore ad_f{}^{n-1}g = \sum_{k=n-i-1}^{n-2} \alpha_k ad_f{}^k g$$
$$\therefore \nabla z_1 ad_f{}^{n-1}g = \sum_{k=n-i-1}^{n-2} \alpha_k \nabla z_1 ad_f{}^k g = 0 \qquad (12)$$

$\therefore$ Contradicts with (11).

- The second property is that $\exists$ a scalar fcn z_1 that satisfy $n-1$ PDEs $\nabla z_1 ad_f{}^k g = 0$
- $\therefore$ From the necessity part of Frobenius theorem, we conclude that the set of vector field must be involutive.
- **Sufficient condition**
  - Involutivity condition $\Longrightarrow$ Frobenius theorem, $\exists$ a scalar fcn $z_1(x)$:

$$L_g z_1 = L_{ad_f g} z_1 = \dots L_{ad_f{}^k g} z_1 = 0, \quad \text{implying}$$
$$L_g z_1 = L_g L_f z_1 = \dots L_g L_f{}^k z_1 = 0$$

- Define the new sets of variable as $z = [z_1 \; L_f z_1 \; \dots \; L_f{}^{n-1} z_1]^T$, to get

$$\begin{aligned} \dot{z}_k &= z_{k+1} \qquad\qquad k = 1, \; \dots, \; n-1 \\ \dot{z}_n &= L_f{}^n z_1 + L_g L_f{}^{n-1} z_1 u \end{aligned} \tag{13}$$

The question is whether $L_g L_f{}^{n-1} z_1$ can be equal to zero.

- Since $\{g, \; ad_f \; g, \; \dots, \; ad_f{}^{n-1} g\}$ are linearly independent in Ω:

$$L_g L_f{}^{n-1} z_1 = (-1)^{n-1} L_{ad_f{}^{n-1} g} z_1$$

- We must have $L_{ad_f{}^{n-1} g} z_1 \neq 0$, otherwise the nonzero vector ∇z_1 satisfies

$$\nabla z_1 \, [g, \; ad_f \; g, \; \dots, \; ad_f{}^{n-1} g] = 0$$

i.e. ∇z_1 is normal to n linearly independent vector $\Longrightarrow$ impossible

- Now, we have:

$$\dot{z}_n = L_f{}^n z_1 + L_g L_f^{n-1} z_1 u = a(x) + b(x) u$$

- Now, select $u = \frac{1}{b(x)}(-a(x) + v)$ to get:

$$\dot{z}_n = v$$

implying input-state linearization is obtained. □

- **Summary: how to perform input-state Linearization**
  1. Construct the vector fields $g,\ ad_f g,\ \dots\ ad_f^{\ n-1} g$
  2. Check the controllability and involutivity conditions
  3. If the conditions hold, obtain the first state z_1 from:

$$\begin{aligned} \nabla z_1 ad_f^{\ i} g &= 0 \quad i = 0,\ \dots,\ n-2 \\ \nabla z_1 ad_f^{\ n-1} g &\neq 0 \end{aligned}$$

  4. Compute the state transformation $z(x) = [z_1\ L_f z_1\ \dots\ L_f^{\ n-1} z_1]^T$ and the input transformation $u = \alpha(x) + \beta(x)v$:

$$\alpha(x) = -\frac{L_f^{\ n} z_1}{L_g L_f^{\ n-1} z_1}$$

$$\beta(x) = \frac{1}{L_g L_f^{\ n-1} z_1}$$

Example:A single-link flexible-joint manipulator:

- The link is connected to the motor shaft via a torsional spring
- **Equations of motion:**

$$I\ddot{q}_1 + MgLsinq_1 + K(q_1 - q_2) = 0$$
$$J\ddot{q}_2 - K(q_1 - q_2) = u$$

- nonlinearities appear in the first equation and torque is in the second equation
- Let:

$$x = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}, \quad f = \begin{bmatrix} x_2 \\ -\frac{MgL}{I} sinx_1 - \frac{K}{I}(x_1 - x_3) \\ x_4 \\ \frac{K}{J}(x_1 - x_3) \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix}$$

- **Controllability and involutivity conditions:**

$$[g \;\; ad_f g \;\; ad_f{}^2 g \;\; ad_f{}^3 g] = \begin{bmatrix} 0 & 0 & 0 & -\frac{K}{IJ} \\ 0 & 0 & \frac{K}{IJ} & 0 \\ 0 & -\frac{1}{J} & 0 & -\frac{K}{J^2} \\ -\frac{1}{J} & 0 & -\frac{K}{J^2} & 0 \end{bmatrix}$$

Example: Cont'd

- It's full rank for $k > 0$ and $IJ < \infty \implies$ vector fields are linearly independent
- Vector fields are constant $\implies$ involutive
- The system is input-state linearizable
- **Computing** $z = z(x), \quad u = \alpha(x) + \beta(x)v$
- $\frac{\partial z_1}{\partial x_2} = 0, \ \frac{\partial z_1}{\partial x_3} = 0, \ \frac{\partial z_1}{\partial x_4} = 0, \ \frac{\partial z_1}{\partial x_1} \neq 0$
- Hence, z_1 is the fcn of x_1 only. Let $z_1 = x_1$, then

$$\begin{aligned} z_2 &= \nabla z_1 f = x_2 \\ z_3 &= \nabla z_2 f = -\frac{MgL}{I} sinx_1 - \frac{K}{I}(x_1 - x_3) \\ z_4 &= \nabla z_3 f = -\frac{MgL}{I} x_2 cosx_1 - \frac{K}{I}(x_2 - x_4) \end{aligned}$$

Example: Cont'd

- The input transformation is given by:

$$
\begin{aligned}
u &= (v - \nabla z_4 f)/(\nabla z_4 g) = \frac{IJ}{K}(v - a(x)) \\
a(x) &= \frac{MgL}{I} sinx_1 (x_2^2 + \frac{MgL}{I} cosx_1 + \frac{K}{I}) \\
&+ \frac{K}{I}(x_1 - x_3)(\frac{K}{I} + \frac{K}{J} + \frac{MgL}{I} cosx_1)
\end{aligned}
$$

- As a result, we get the following set of linear equations

$$
\begin{aligned}
\dot{z}_1 &= z_2, \quad \dot{z}_2 = z_3 \\
\dot{z}_3 &= z_4, \quad \dot{z}_4 = v
\end{aligned}
$$

- The inverse of the state transformation is given by:

$$
\begin{aligned}
x_1 &= z_1, \quad x_2 = z_2 \\
x_3 &= z_1 + \frac{I}{K}\left(z_3 + \frac{MgL}{I} sinz_1\right) \\
x_4 &= z_2 + \frac{I}{K}\left(z_4 + \frac{MgL}{I} z_2 cosz_1\right)
\end{aligned}
$$

Input-State Linearization

- State and input transformations are defined globally
- In this example, transformed state have physical meaning, z_1 : link position, z_2 : link velocity, z_3 : link acceleration, z_4 : link jerk.
- It could be obtained by I-O linearization, i.e. by differentiating the output q_1.
- We can transform the inequality (11) to a normalized equation by setting $\nabla z_1 ad_f{}^{n-1} g = 1$ resulting in:

$$[ad_f{}^0 g \;\; ad_f{}^1 g \;\; \ldots \;\; ad_f{}^{n-2} g \;\; ad_f{}^{n-1} g\,] \begin{bmatrix} \frac{\partial z_1}{\partial x_1} \\ \vdots \\ \frac{\partial z_1}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Control Design

- Once, the linearized dynamics is obtained, either a tracking or stabilization problem can be solved
- For instance, in flexible-joint manipulator case, we have

$$z_1^{(4)} = v$$

- Then, a tracking controller can be obtained as

$$v = z_{d1}^{(4)} - a_3\tilde{z}_1^{(3)} - a_2\ddot{\tilde{z}}_1 - a_1\dot{\tilde{z}}_1 - a_0\tilde{z}_1$$

where $\tilde{z}_1 = z_1 - z_{d1}$.

- The error dynamics is then given by:

$$\tilde{z}_1^{(4)} + a_3\tilde{z}_1^{(3)} + a_2\ddot{\tilde{z}}_1 + a_1\dot{\tilde{z}}_1 + a_0\tilde{z}_1 = 0$$

- The above dynamics is exponentially stable if a_i are selected s.t.

$$s_4 + a_3s^3 + +a_2s^2 + a_1s + a_0 \quad \text{is Hurwitz}$$

Input-Output Linearization

- Consider the system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{14}$$

- Input-output linearization yields a linear relationship between the input output y and the input v (similar to v in I-S Lin.)
  - How to generate a linear I-O relation for such systems?
  - What are the internal dynamics and zero-dynamics associated with this I-O linearization
  - How to design a stable controller based on this technique?
- **Performing I-O Linearization**
  - The basic approach is to differentiate the output y until the input u appears, then design u to cancel nonlinearities
  - Sometime, cancelation might not be possible due to the undefined relative degree.

Well Defined Relative Degree

- Differentiate y and express it in the form of Lie derivative:

$$\dot{y} = \nabla h(f + gu) = L_f h(x) + L_g h(x) u$$

  if $L_g h(x) \neq 0$ for some $x = x_0$ in Ω_x, then continuity implies that $L_g h(x) \neq 0$ in some neighborhood Ω of x_0. Then, the input transformation

$$u = \frac{1}{L_g h(x)}(-L_f h(x) + v)$$

  results in a linear relationship between y and v, namely $\dot{y} = v$.

- If $L_g h(x) = 0$ for all $x \in \Omega_x$, differentiate $\dot{y}$ to obtain

$$\ddot{y} = L_f{}^2 h(x) + L_g L_f h(x) u$$

- If $L_g L_f h(x) = 0$ for all $x \in \Omega_x$, keep differentiating until **for some integer** r, $L_g L_f{}^{r-1} h(x) \neq 0$ for some $x = x_0 \in \Omega_x$

- Hence, we have

$$y^{(r)} = L_f{}^r h(x) + L_g L_f{}^{r-1} h(x) u \tag{15}$$

and the control law

$$u = \frac{1}{L_g L_f{}^{r-1} h(x)} (-L_f{}^r h(x) + v)$$

yields a linear mapping:

$$y^{(r)} = v$$

- The number r of differentiation required for u to appear is called the relative degree of the system.
- $r \leq n$, if $r = n$, the input-state realization is obtained with $z_1 = y$.
- **Definition:** *The SISO system is said to have a relative degree r in Ω if:*

$$\begin{aligned} L_g L_f{}^i h(x) &= 0 \qquad \mathbf{0} \leq i \leq r-2 \\ L_g L_f{}^{r-1} h(x) &\neq 0 \end{aligned}$$

Undefined Relative Degree

- Sometimes, we are interested in the properties of a system about a specific operating point x_0.
- Then, we say the system has relative degree r at x_0 if

$$L_g L_f{}^{r-1} h(x_0) \neq 0$$

- However, it might happen that $L_g L_f{}^{r-1} h(x)$ is zero at x_0, but nonzero in a close neighborhood of x_0.
- The relative degree of the nonlinear system is then undefined at x_0.
- **Example:**

$$\ddot{x} = \rho(x, \dot{x}) + u$$

where ρ is a smooth nonlinear fcn. Define $x = [x \ \dot{x}]^T$ and let $y = x \implies$ the system is in companion form with $r = 2$.

- However, if we define $y = x^2$, then:

$$\begin{aligned} \dot{y} &= 2x\dot{x} \\ \ddot{y} &= 2x\ddot{x} + 2\dot{x}^2 = 2x\rho(x,\dot{x}) + 2xu + 2\dot{x}^2 \quad \Longrightarrow \\ L_g L_f h &= 2x \end{aligned} \tag{16}$$

- The system has neither relative degree 1 nor 2 at x_0.
- Sometime, change of output leads us to a solvable problem.
- We assume that the relative degree is well defined.
- **Normal Forms**
  - When, the relative degree is defined as $r \leq n$, using $y,\ \dot{y},\ \ldots,\ y^{(r-1)}$, we can transform the system into the so-called normal form.
  - Norm form allows a formal treatment of the notion of internal dynamics and zero dynamics.
  - Let $$\mu = [\mu_1\ \mu_2\ \ldots\ \mu_r]^T \ = \ [y\ \dot{y}\ \ldots\ y^{(r-1)}]^T$$

    in a neighborhood Ω of a point x_0.

Normal Form

- The normal form of the system can be written as

$$\dot{\mu} = \begin{bmatrix} \mu_2 \\ \vdots \\ \mu_r \\ a(\mu, \Psi) + b(\mu, \Psi)u \end{bmatrix} \tag{17}$$

$$\dot{\Psi} = w(\mu, \Psi) \tag{18}$$

$$y = \mu_1$$

- The μ_i and Ψ_j are called *normal coordinate* or *normal states*.
- The first part of the Normal form, (17) is another form of (15), however in (18) the input u does not appear.
- The system can be transformed to this form if the state transformation $\phi(x)$ is a local diffeomorphism: $\phi(\mu_1 \ldots \mu_r \; \Psi_1 \ldots \Psi_{n-r})^T$
- To show that ϕ is a diffeomorphism, we must show that the Jacobian is invertible, i.e. $\nabla\mu_i$ and $\nabla\Psi_i$ are all linearly independent.

Normal Form

- $\nabla\mu_i$ are linearly independent $\Longrightarrow$ μ can be part of state variables, (μ is output and its $r-1$ derivatives)
- There exist $n-r$ other vector fields that complete the transformation
- Note that u does not appear in (17), hence:

$$\nabla\Psi_j g = 0 \quad 1 \leq j \leq n-r$$

$\therefore$ Ψ can be obtained by solving $n-r$ PDE above.

- Generally, internal dynamics can be obtained simpler by intuition.
- **Zero Dynamics**
  - System dynamics into two parts:
    1. external dynamics $\dot{\mu}$
    2. internal dynamics $\dot{\Psi}$
  - For tracking problems ($y \longrightarrow y_d$), one can easily design v once the linear relation is obtained.
  - The question is whether the internal dynamics remain bounded

Zero-Dynamics

- Stability of the **zero dynamics** (i.e. internal dynamics when y is kept 0) gives an idea about the stability of internal dynamics
- u is selected s.t. y remains zero at all time.

$$\begin{aligned} y^{(r)}(t) &= L_f{}^r h(x) + L_g L_f{}^{r-1} h(x) u_0 \equiv 0 \Longrightarrow \\ u_0(x) &= \frac{-L_f{}^r h(x)}{L_g L_f{}^{r-1} h(x)} \end{aligned}$$

- $\therefore$ In normal form:

$$\begin{cases} \dot{\mu} = 0 \\ \dot{\Psi} = w(0, \Psi) \end{cases}$$

$$u_0(\Psi) = \frac{-a(0, \Psi)}{b(0, \Psi)} \tag{19}$$

Example

- Consider

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -x_1 \\ 2x_1x_2 + sinx_2 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} e^{2x_2} \\ 1/2 \\ 0 \end{bmatrix} u \\
y &= h(x) = x_3
\end{aligned}
$$

- We have

$$
\begin{aligned}
\dot{y} &= 2x_2 \\
\ddot{y} &= 2\dot{x}_2 = 2(2x_1x_2 + sinx_2) + u
\end{aligned}
$$

- The system has relative degree $r = 2$ and

$$
\begin{aligned}
L_f\ h(x) &= 2x_2 \\
L_g\ h(x) &= 0 \\
L_gL_f\ h(x) &= 1
\end{aligned}
$$

Example Cont'd

- To obtain the normal form

$$\begin{aligned} \mu_1 &= h(x) = x_3 \\ \mu_2 &= L_f\ h(x) = 2x_2 \end{aligned}$$

- The third function $\Psi(x)$ is obtained by

$$L_g\Psi = \frac{\partial \Psi}{\partial x_1} e^{2x_2} + \frac{1}{2}\frac{\partial \Psi}{\partial x_2} = 0$$

- One solution is $\Psi(x) = 1 + x_1 - e^{2x_2}$
- Consider the jacobian of state transformation $z = [\mu_1\ \mu_2\ \Psi]^T$. The Jacobian matrix is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & -2e^{2x_2} & 0 \end{bmatrix}$$

Example Cont'd

- The Jacobian is non-singular for any x. In fact, inverse transformation is given by:

$$\begin{aligned} x_1 &= -1 + \Psi + e^{\mu_2} \\ x_2 &= \frac{1}{2}\mu_2 \\ x_3 &= \mu_1 \end{aligned}$$

- State transformation is valid globally and the normal form is given by:

$$\begin{aligned} \dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= 2(-1 + \Psi + e^{\mu_2})\mu_2 + 2sin(\mu_2/2) + u \\ \dot{\Psi} &= (1 - \Psi - e^{\mu_2})(1 + 2\mu_2 e^{\mu_2}) - 2sin(\mu_2/2)e^{\mu_2}) \end{aligned} \tag{20}$$

- Zero dynamics is obtained by setting $\mu_1 = \mu_2 = 0 \implies$

$$\dot{\Psi} = -\Psi \tag{21}$$

Zero-Dynamics

- In order to obtain the zero dynamics, it is not necessary to put the system into normal form
- since μ is known, we can intuitively find $n - r$ vector to complete the transformation.
- As mention before, zero dynamics is obtained by substituting u_0 for u in internal dynamics.
- **Definition:** *A nonlinear system with asymptotically stable zero dynamics is called asymptotically* **minimum phase**
- If the zero dynamics is stable for all x, the system is globally minimum phase, otherwise the results are local.

Local Asymptotic Stabilization

- Consider again the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{22}$$

Assume that the system is I-O linearized, i.e.

$$y^{(r)} = L_f{}^r h(x) + L_g L_f{}^{r-1} h(x) u \tag{23}$$

and the control law

$$u = \frac{1}{L_g L_f{}^{r-1} h(x)} (-L_f{}^r h(x) + v) \tag{24}$$

yields a linear mapping:

$$y^{(r)} = v$$

- Now let v be chosen as

$$v = -k_{r-1} y^{(r-1)} - \ \dots \ - k_1 \dot{y} - k_0 y \tag{25}$$

where k_i are selected s.t. $K(s) = s^r + k_{r-1} s^{r-1} + \ \dots \ + k_1 s + k_0$ is Hurwitz

- Then, provided that the zero-dynamics is asymptotically stable, the control law (24) and (25) locally stabilize the whole system:
- **Theorem:** *Suppose the nonlinear system (22) has a well defined relative degree r and its associated zero-dynamics is locally asymptotically stable. Now, if k_i are selected s.t. $K(s) = s^r + k_{r-1}s^{r-1} + \ \dots \ + k_1 s + k_0$ is Hurwitz, then the control law (24) and (25) yields a locally asymptotically stable system.*
- **Proof:** First, write the closed-loop system in a normal form:

$$\dot{\mu} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ -k_0 & -k_1 & -k_2 & \dots & -k_{r-1} \end{bmatrix} = A\mu$$

$$\dot{\Psi} = w(\mu, \Psi) = A_1\mu + A_2\Psi + h.o.t.$$

The above Eq. can be written as:

$$\frac{d}{dt}\begin{bmatrix} \mu \\ \Psi \end{bmatrix} = \begin{bmatrix} A & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} \mu \\ \Psi \end{bmatrix} + h.o.t.$$

- Now, since the zero dynamics is asymptotically stable, its linearization $\dot{\Psi} = A_2\Psi$ is either asymptotically stable or marginally stable.
  - If A_2 is **asymptotically stable**, then all eigenvalues of the above system matrix are in LHP and the linearized system is stable and the nonlinear system is locally asymptotically stable
  - If A_2 is **marginally stable**, asymptotic stability of the closed-loop system was shown in (Byrnes and Isidori, 1988).
- For stabilization where state convergence is required, we can freely choose $y = h(x)$ to affect zero-dynamics.
- **Example:** Consider the nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_1^2 x_2 \\ \dot{x}_2 &= 3x_2 + u \end{aligned}$$

- System linearization at $x = 0$:

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= 3x_2 + u \end{aligned}$$

thus has an uncontrollable mode

- **Example (cont'd)**
  - Define $y = -2x_1 - x_2 \implies$

$$\dot{y} = -2\dot{x}_1 - \dot{x}_2 = -2x_1^2 x_2 - 3x_2 - u$$

  - Hence, the relative degree $r = 1$ and the associated zero-dynamics is

$$\dot{x}_1 = -2x_1^3$$

  - The zero-dynamics is asymptotically stable, hence the control law $u = -2x_1^2 x_2 - 4x_2 - 2x_1$ locally stabilizes the system
- **Global Asymptotic Stabilization**
- Stability of the zero-dynamics only guarantees local stability unless relative degree is n in which case there is no internal dynamics
- Can the idea of I-O linearization be used for **global stabilization** problem?
- Can the idea of I-O linearization be used for systems with **unstable zero dynamics**?

Global Asymptotic Stabilization

- Global stabilization approach based on partial feedback linearization is to simply regard the problem as a standard Lyapunov controller design problem
- **But** simplified by the fact that in *normal form* part of the system dynamics is now linear.
- The basic idea is to view μ as the input to the internal dynamics and Ψ as its output.
  - The first step: find the control law $\mu_0 = \mu_0(\Psi)$ which stabilizes the internal dynamics with the corresponding Lyapunov fcn V_0.
  - Then: find a Lyapunov fcn candidate for the whole system (as a modified version of V_0) and choose the control input v s.t. V be a Lyapunov fcn for the whole closed-loop dynamics.

Example:

- Consider a nonlinear system with the normal form:

$$\begin{aligned} \dot{y} &= v \\ \ddot{z} &+ \dot{z}^3 + yz = 0 \end{aligned} \tag{26}$$

where v is the control input and $\Psi = [z\ \dot{z}]^T$

- Considering y as an input to internal dynamics (26), it would be asymptotically stabilized by the choice of $y = y_0 = z^2$
  - Let V_0 be a Lyap. fcn:

$$V_0 = \frac{1}{2}\dot{z}^2 + \frac{1}{4}z^4$$

  - Differentiating V_0 along the actual dynamics results in

$$\dot{V}_0 = -\dot{z}^4 - z\dot{z}(y - z^2)$$

Example Cont'd

- Consider the Lyap. fcn candidate, obtained by adding a quadratic “error” term in $y - y_0$ to V_0

$$\begin{aligned} V &= V_0 + \frac{1}{2}(y - z^2)^2 \\ \therefore \dot{V} &= -\dot{z}^4 - (y - z^2)(v - 3z\dot{z}) \end{aligned}$$

- The following choice of control action will then make $\dot{V}$ **n.s.d.**

$$v = -y + z^2 + 3z\dot{z}$$

$$\therefore \dot{V} = -\dot{z}^4 - (y - z^2)^2$$

- Application of Invariant-set theorem shows all states converges to zero

Example: A non-minimum phase system

- Consider the system dynamics
$$\begin{aligned}\dot{y} &= v\\ \ddot{z} &+ \dot{z}^3 - z^5 + yz = 0\end{aligned}$$
where again $\Psi = [z\ \dot{z}]^T$
  - The system is non-minimum phase since its zero-dynamics is unstable
  - The zero-dynamics would be stable if we select $y = 2z^4$:
$$V_0 = \frac{1}{2}\dot{z}^2 + \frac{1}{6}z^6 \rightsquigarrow \dot{V}_0 = -\dot{z}^4 - z\dot{z}(y - 2z^4)$$
  - Consider the Lyap. fcn candidate
$$V = V_0 + \frac{1}{2}(y - 2z^4)^2 \rightsquigarrow \dot{V} = -\dot{z}^4 - (y - 2z^4)(v - 8z^3\dot{z} - z\dot{z})$$
  - suggesting the following choice of control law
$$v = -y + 2z^4 + 8z^3\dot{z} + z\dot{z} \rightsquigarrow \dot{V} = -\dot{z}^4 - (y - 2z^4)^2$$
  - Application of Invariant-set theorem shows all states converges to zero

Tracking Control

- I/O linearization can be used in tracking problem
- Let $\mu_d = [y_d \ \dot{y}_d \ \ldots \ y_d^{(r-1)})]^T$ and the tracking error $\tilde{\mu}(t) = \mu(t) - \mu_d(t)$
- **Theorem:** *Assume the system (22) has a well defined relative degree r and μ_d is smooth and bounded and that the solution Ψ_d:*

$$\dot{\Psi}_d = w(\mu_d, \Psi_d), \quad \Psi_d(0) = 0$$

exists and bounded and is uniformly asymptotically stable. Choose k_i s.t $K(s) = s^r + k_{r-1}s^{r-1} + \ \ldots \ + k_1 s + k_0$ is Hurwitz, then by using

$$u = \frac{1}{L_g L_f{}^{r-1}\mu_1}[-L_f{}^r\mu_1 + y_d^{(r)} - k_{r-1}\tilde{\mu}_r - \ \ldots \ - k_0\tilde{\mu}_r] \qquad (27)$$

the whole system remains bounded and the tracking error $\tilde{\mu}$ converge to zero exponentially.

- **Proof:** Refer to Isidori (1989).
- For perfect tracking $\mu(0) \equiv \mu_d(0)$

Tracking Control for Non-minimum Phase Systems:

- The tracking control (27) cannot be applied to non-minimum phase systems since they cannot be inverted
- Hence we cannot have perfect or asymptotic tracking and should seek controllers that yields small tracking errors
- One approach is the so-called Output redefinition
  - The new output y_1 is defined s.t. the associated zero-dynamics is stable
  - y_1 is defined s.t. it is close to the original output y in the frequency range of interest
  - Then, tracking y_1 also implies good tracking the original output y
- **Example:** Consider a linear system

$$y = \frac{\left(1 - \frac{s}{b}\right) B_0(s)}{A(s)} u \quad b > 0$$

- Perfect/asymptotic tracking is impossible due to the presence of zero @ $s = b$

Example Cont'd

- Let us redefine the output as

$$y_1 = \frac{B_0(s)}{A(s)} u$$

  with the desired output for y_1 be simply y_d

- A controller can be found s.t. y_1 asymptotically tracks y_d. What about the actual tracking error?

$$y(s) = \left(1 - \frac{s}{b}\right) y_1 = \left(1 - \frac{s}{b}\right) y_d$$

- Thus, the tracking error is proportional to the desired velocity $\dot{y}_d$:

$$y(t) - y_d(t) = -\frac{\dot{y}_d(t)}{b}$$

- $\therefore$ Tracking error is bounded as long as $\dot{y}_d$ is bounded, it is small when the frequency content of y_d is well below b

Example Cont'd

- An alternative output, motivated by $(1-\frac{s}{b}) \approx 1/(1+\frac{s}{b})$ for small $|s|/b$:

$$y_2 = \frac{B_0(s)}{A(s)(1+\frac{s}{b})} u$$

$$y(s) = \left(1-\frac{s}{b}\right)\left(1+\frac{s}{b}\right) y_d = \left(1-\frac{s^2}{b^2}\right) y_d$$

- Thus, the tracking error is proportional to the desired acceleration $\ddot{y}_d$:

$$y(t) - y_d(t) = -\frac{\ddot{y}_d(t)}{b^2}$$

- Small tracking error if the frequency content of y_d is below b

Tracking Control

- Another approximate tracking (Hauser, 1989) can be obtained by
  - When performing I/O linearization, using successive differentiation, simply **neglect** the terms containing the input
  - Keep differentiating n timed (system order)
  - Approximately, there is no zero dynamics
  - It is meaningful if the coefficients of u at the intermediate steps are "small" or the system is "weakly non-minimum phase" system
  - The approach is similar to neglecting fast RHP zeros in linear systems.
- Zero-dynamics is the property of the plant, choice of input and output and cannot be changed by feedback:
  - Modify the plant (distribution of control surface on an aircraft or the mass and stiffness in a robot)
  - Change the output (or the location of sensor)
  - Change the input (or the location of actuator)