Signals and Systems
Lecture 8: State Space Representation

Farzaneh Abdollahi

Department of Electrical Engineering

Amirkabir University of Technology

Winter 2014

State Space Representation
- Block Diagram of State Space Representation
- Solving State Equations by UL
- Method to Find Transition Matrix
- Defining Transfer Function from State Space Eq.
- State Space Realizations

State Space Representation

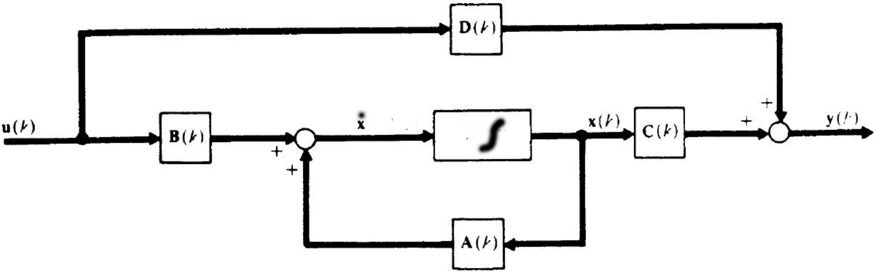
- Previously we learnt that for a LTI system with $y(t)$: output signal, $u(t)$: input signal, and $h(t)$ impulse response
  $y(t) = h(t) * u(t) \rightsquigarrow Y(s) = H(s)U(s)$
- This representation of the system only express I/O relation
- It does not give us internal specification of the system.
- State space representation not only provide us information on I/O but also gives us good view on internal specification of the system
- States of a system at time t_0 includes min required information to express the system situation at time t_0
- They are first degree equations

- State space representation of LTI system

$$\dot{X}(t) = AX(t) + BU(t) \text{ state equations}$$
$$Y(t) = CX(t) + DU(t) \text{ output equations}$$

  - $X \in R^n$: state vector
  - $U \in R^m$: input vector
  - $Y \in R^p$ output vector
  - $A^{n \times n}$: System Matrix
  - $B^{n \times m}$: input matrix
  - $C^{p \times n}$: output matrix
  - $D^{p \times m}$: coupling matrix
- Number of state usually equals to degree of the system
  - It usually equals to number of active elements in the system (# of capacitors and inductors in RLC circuits)
  - However in some cases like having cut-set of inductors and loop of capacitors degree of the system would be less than # of active elements
  - One could choose number of the states greater than n in such case some modes are not observable or controllable
- Set of states is not unique for a system

Block Diagram of State Space Representation



Solving State Equations by UL

- Assuming x is causal $\rightsquigarrow$ we are using UL
- $\dot{x} = Ax + Bu \overset{UL}{\Leftrightarrow} sX(s) - x(0) = AX(s) + BU(s)$
- $X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$
- Let us define $\phi(t) = L^{-1}\{(sI - A)^{-1}\}$: Transition Matrix
- $x(t) = \underbrace{\phi(t)x(0)}_{ZIR} + \underbrace{\int_0^t \phi(\tau)Bu(t-\tau)d\tau}_{ZSR}$
- For LTI systems $\phi(t) = e^{At}$

Methods to Find Transition Matrix

1. $\phi(t) = L^{-1}(sI - A)^{-1}$
   - Example: $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$
   - $\phi(t) = L^{-1}(sI - A)^{-1} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$
   - For large A, finding inverse matrix is time consuming and complicated

Methods to Find Transition Matrix

2. Approximate by Infinite Power Series

- The transition matrix is system specification and input does not affect on it:

$$\dot{x} = Ax(t) \quad (1)$$
$$x(t) = \Phi(t)x(0) \quad (2)$$

- Let us represent transition matrix by an infinite power series:

$$x(t) = (k_0 + k_1 t + k_2 t^2 + \ldots)x_0 \quad (3)$$

- $\dot{x}(t) = (k_1 + 2k_2 t + \ldots)x_0$
- $\therefore (k_1 + 2k_2 t + 3k_3 t^2 + \ldots)x_0 = A(k_0 + k_1 t + \ldots)x_0$
- $k_1 = Ak_0,\ k_2 = A\frac{k_1}{2},\ k_3 = A\frac{k_2}{3}$
- Substitute $t = 0$ in (3): $k_0 = I$
- $k_0 = I,\ k_1 = A,\ k_2 = \frac{A^2}{2!},\ k_3 = \frac{A^3}{3!}$
- $\phi(t) = e^{At} = I + At + A^2\frac{t^2}{2!} + \ldots$

Methods to Find Transition Matrix

3. By Cayley Hamilton Theorem

- Reminder: Eigne Value of Matrix A is a scalar value λ s.t.
  - $Av = \lambda v$
  - where v is a vector named Eigne vector
- To find eigne values:
  - $|\lambda I - A| = 0 \rightsquigarrow \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$
  - The above equation is named characteristic equation of matrix A
- Considering Cayley Hamilton Theorem result in [1]:
  $e^{At} = a_0(t)I + a_1(t)A + \ldots + a_{n-1}(t)A^{n-1}$
- Eigne vector of matrix A is eigne vector ofe^{At}

$$\left.\begin{array}{r} Av_i = \lambda_i v_i \\ A^2 v_i = \lambda_i^2 v_i \\ \vdots \\ A^n v_i = \lambda_i^n v_i \end{array}\right\} \Rightarrow e^{\lambda_i t} v_i = (a_0(t)I + a_1(t)\lambda_i + a_2(t)\lambda_i^2 + \ldots + a_{n-1}(t)\lambda^{n-1})v_i$$

- By assuming n distinct eigne values and solving n equations all coefficients $a_i(t)$ are obtained

Example

- $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$
- $\lambda_1 = -2, \ \lambda_2 = -3$
- $e^{-2t} = a_0(t) - 2a_1(t)$
- $e^{-3t} = a_0(t) - 3a_1(t)$
- $a_1(t) = e^{-2t} - e^{-3t}$
- $a_0(t) = 3e^{-2t} - 2e^{-3t}$
- $\phi(t) = e^{At} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$

Defining Transfer Function from State Space Eq.

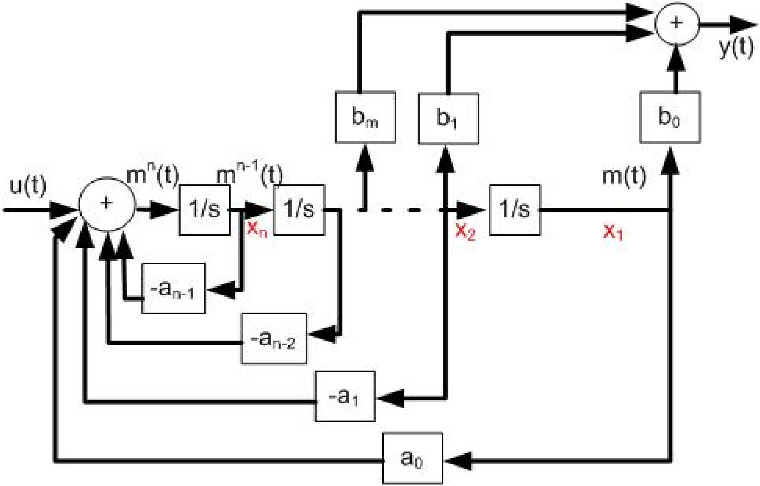
$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

- Transfer fcn: $H(s) = \frac{Y(s)}{U(s)}$
- Transfer fcn is ZSR:
  $sX(s) = AX(s) + BU(s) \rightsquigarrow X(s) = (sI - A)^{-1}BU(s)$
- $Y(s) = [C(sI - A)^{-1}B + D]U(s)$
- $H(s) = C(sI - A)^{-1}B + D = C\frac{adj(sI-A)}{det(sI-A)}B + D$
- Poles of a system are eigne values of matrix A
- BUT all eigne values of A are not poles of the system (due to zero-pole cancelation)
  - If an unstable poles is canceled by a zero the system is not internally stable anymore

State Space Realizations

- Several state space realization can be obtained from a transfer fcn. two of them are introduced here.
  1. Controllable Canonical Form
     - Consider $H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} = \frac{b(s)}{a(s)}$, $n > m$
     - If $n = m$ then we can define $H(s) = b_n + \frac{\bar{b}_m s^m + \bar{b}_{m-1}s^{m-1} + \ldots + \bar{b}_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}$
     - Let us define a axillary fcn $M(s)$
     - $\frac{Y(s)}{U(s)} = \frac{Y(s)}{M(s)} \cdot \frac{M(s)}{U(s)} = b(s) . \frac{1}{a(s)}$
     - $M(s)a(s) = U(s) \rightsquigarrow M(s)(s^n + a_{n-1}s^{n-1} + \ldots + a_0) = U(s)$
     - $m^n(t) = -a_{n-1}m^{n-1}(t) - \ldots - a_0 m(t) + u(t)$
     - $Y(s) = b(s)M(s) \rightsquigarrow y(t) = b_m m(t)^m + \ldots + b_1 \dot{m}(t) + b_0 m(t)$

Controllable Canonical Form



- (assume $m = n - 1$) By defining output of integrators as states:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n + u \\
y &= b_0 x_1 + b_1 x_2 + \ldots + b_{n-1} x_n
\end{aligned}
$$

- $\therefore A = \begin{bmatrix} 0 & 1 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & 1 \\ -a_0 & -a_1 & \ldots & \ldots & -a_{n-1} \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix},$

  $C = [b_0 \ b_1 \ \ldots \ \ldots \ b_{n-1}], D = 0$

- **Example:** $H(s) = \frac{s^3+6s^2+5s+2}{s^3+7s^2+3s+1}$

2. Diagonal Form and Jordan Form

- Consider characteristic equation has n separate roots: $H(s) = \frac{\beta_1}{s-P_1} + \frac{\beta_2}{s-P_2} + \frac{\beta_3}{s-P_3} + \ldots + \frac{\beta_n}{s-P_n}$
- $\therefore A = \begin{bmatrix} P_1 & \ldots & \ldots & 0 \\ 0 & P_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & P_n \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$
  $C = [\beta_1 \; \beta_2 \; \ldots \; \beta_n], D = 0$

2. Diagonal Form and Jordan Form

- If there are frequent poles, for example if there are three similar poles : $H(s) = \frac{\beta_1}{s-P_1} + \frac{\beta_2}{(s-P_2)^3} + \frac{\beta_3}{(s-P_2)^2} + \frac{\beta_4}{s-P_2}$, matrices A, B, and C are modified as follows:

$$A = \begin{bmatrix} P_1 & 0 & 0 & 0 \\ 0 & P_2 & 1 & 0 \\ 0 & 0 & P_2 & 1 \\ 0 & 0 & 0 & P_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = [\beta_1 \ \beta_2 \ \beta_3 \ \beta_4], D = 0$$

- **Example:** $H(s) = \frac{s^2+3s+1}{(s+1)^2(s+3)}$

W. L. Brogan, *Modern Control Theory (3rd Edition)*. Prentice Hall, 1991.