Nonlinear Control
Lecture 9: Sliding Control

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Sliding Control
- Sliding Surface
- Integral Control
- Gain Margins

Continuous Approximations of Switching Control Laws

Sliding Control

- Sliding control is a robust control technique to control systems with model imprecision and uncertainties.
- Sliding control is based on the idea that "controlling a 1^{st} order system is much easier than the general n^{th} order system
- To achieve this goal:
  1. A first order system (sliding surface) is proposed and provide a condition (sliding condition) to make the introduced surface an invariant set of the system stability
  2. A control is designed to *reach* to the sliding surface
- Providing perfect performance in presence of arbitrary parameter inaccuracy is at the price of extremely high control activity.
- $\therefore$ a modification of control law is required to provide an effective trade-off between tracking performance and parametric uncertainty.
- In some specific applications, such as those involving the control of electric motor the unmodified control law can be applied directly.

Sliding Surface

- Consider single input dynamics
$$x^{(n)} = f(x) + b(x)u \qquad (1)$$
  - f is not exactly known, upper bounded by known continuous function of x
  - b is not exactly known, its sign is known and upper bounded by known continuous function of x
- Objective: find u, s.t. $\mathbf{x}$ track $\mathbf{x}_d = [x_d, \dot{x}_d, \ldots, x_d^{(n-1)}]^T$ in presence of imprecision on $f(x)$ and $b(x)$
- Tracking error vector: $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_d = [\tilde{x}\ \dot{\tilde{x}} \ldots \tilde{x}^{(n-1)}]$
- Define a time-varying surface $S(t)$ in state-space R^n by scaler equation $s(\mathbf{x}; t) = 0$:
$$s(\mathbf{x}; t) = (\frac{d}{dt} + \lambda)^{n-1}\tilde{x} \qquad (2)$$

  where $\lambda > 0$ conts.
  - for $n = 2 \rightsquigarrow s = \dot{\tilde{x}} + \lambda\tilde{x}$, s is a weighted sum of position error and velocity error
  - for $n = 3 \rightsquigarrow s = \ddot{\tilde{x}} + 2\lambda\dot{\tilde{x}} + \lambda^2\tilde{x}$

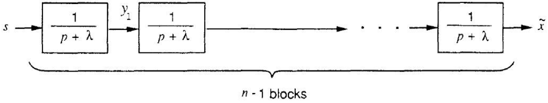
Sliding Surface

- **The problem of tracking the n-dimensional vector x_d (the original tracking problem) can be replaced by a 1st-order stabilization problem in s.**
  - Given initial condition $\mathbf{x}_d(0) = \mathbf{x}(0)$, the problem of tracking $x \equiv x_d$ is equivalent to remaining on the surface $S(t)$ for all $t > 0$ ($s \equiv 0$ represents a linear differential equation whose unique solution is $\tilde{x} \equiv 0$)
  - In (1),s contains $\tilde{x}(n-1) \rightsquigarrow$ we only need to differentiate s once for the input u to appear.
  - Bounds on s can be directly translated into bounds on $\tilde{x} \rightsquigarrow s$ represents a true measure of tracking performance. When $\tilde{\mathbf{x}}(0) = 0$:

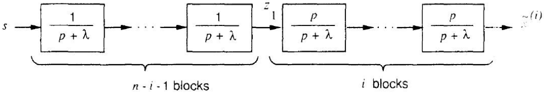
$$\forall t \geq 0, |s(t)| \leq \Phi \Rightarrow \forall t \geq 0, |\tilde{x}^{(i)}| \leq (2\lambda)^i \varepsilon, \quad i = 0, ..., n-1 \tag{3}$$

where $\varepsilon = \Phi / \lambda^{n-1}$

- Proof: $\tilde{x}$ is obtained from s through a sequence of first-order lowpass filters, shown in Fig.



- Let y_1 output of first filter: $y_1 = \int_0^t e^{-\lambda(t-T)} s(T)dT, |s| \leq \Phi \Rightarrow$ $|y_1| \leq \Phi \int_0^t e^{-\lambda(t-T)} dT = (\Phi/\lambda)(1 - e^{-\lambda t}) \leq \Phi/\lambda$
- Repeat the same procedure all the way to $y_{n-1} = \tilde{x} \rightsquigarrow |\tilde{x}| \leq \Phi/\lambda^{n-1} = \varepsilon$
- To obtain $\tilde{x}^{(i)}$, see the Fig b



- The output of the $(n-1-i)^{th}$ filter: $z_1 < \Phi/\lambda^{n-1-i}$
- Note that $\frac{p \pm \lambda}{p+\lambda} = 1 - \frac{\lambda}{\lambda+p} \leq 1 + \frac{\lambda}{\lambda+p}$
- $\therefore |\tilde{x}^{(i)}| \leq (\Phi/\lambda^{n-1-i})(1 + \frac{\lambda}{\lambda})^i = (2\lambda)^i \varepsilon$
- If $\tilde{\mathbf{x}}(0) \neq 0$, $\rightsquigarrow$, (3) is obtained asymptotically, within a short time-constant $(n-1)/\lambda$.

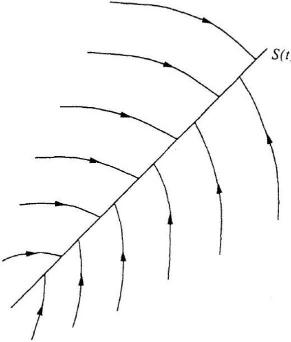
Sliding Condition

- To keep the scalar s at zero, a control law u it should be found s.t outside of $S(t)$:

$$\frac{1}{2}\frac{d}{dt}s^2 \leq -\eta|s| \tag{4}$$

where $\eta > 0$ conts.

- $\therefore$ The squared "distance" to the surface, s^2 , decreases along all system trajectories. ($V = \frac{1}{2}s^2$)
- (4), so-called sliding condition, makes the surface an *invariant set*.
- By keeping the invariant set, some disturbances or dynamic uncertainties can be tolerated.
- $S(t)$ is sliding surface; behavior of the system on the surface is sliding mode

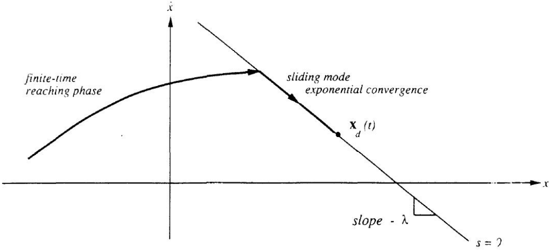


The sliding condition

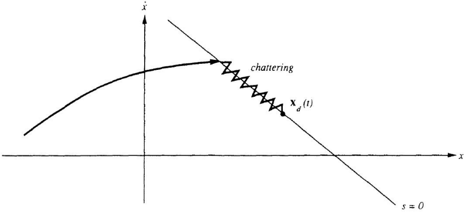
- If it is on the sliding surface, the system behavior can be expressed by $(\frac{d}{dt} + \lambda)^{n-1}\tilde{x} = 0$
- If sliding condition is guaranteed, for *nonzero* initial condition, $(\mathbf{x}(0) \neq \mathbf{x}_d(0))$, the surface $S(t)$ will be reached in a finite time smaller than $|s(t=0)|/\eta$:
  - For t_{reach}: required time to reach $s=0$, integrate (4) from 0 to t_{reach}: $s(t_{reach}) - s(0) = 0 - s(0) < -\eta(t_{reach} - 0) \rightsquigarrow t_{reach} \leq |s(t=0)|/\eta$
- Once on the surface, tracking error tends exponentially to zero with time constant $(n-1)/\lambda$
  - from the sequence of $(n-1)$ filters of time constants equal to $1/\lambda$

Local Asymptotic Stabilization

- For $n = 2$
  - sliding surface is a line with slope $-\lambda$
  - Starting with any initial conditions, the traj. reaches the time-varying surface in finite time $\leq |s(t=0)|/\eta$
  - Then slide along the surface towards x_d exp. with time constant $1/\lambda$



- After defining the sliding surface s, the control is designed in two steps
  1. A feedback control law u is selected so as to verify sliding condition (4)
  2. The discontinuous control law u is suitably smoothed to achieve an optimal trade-off between control bandwidth and tracking precision
     - To cope with modeling imprecision and disturbances, the control law has to be discontinuous across $S(t)$.
     - Implementing the associated control switchings is always imperfect (switching is not instantaneous, and the value of s is not known with infinite precision) $\rightsquigarrow$ yields chattering
     - Chattering $\rightsquigarrow$ high control activity and may excite high frequency dynamics neglected in modeling (such as unmodeled structural modes, neglected time-delays, and so on).



Chattering as a result of imperfect control switchings

Example

- Consider
$$\ddot{x} = f + u \tag{5}$$

- f is unknown,but estimated by $\hat{f}$, estimation error on f assumed to be bounded by known function $F = F(x, \dot{t})$ $|\hat{f} - f| \leq F$

- To track $x \equiv x_d$, define the sliding surface:
$$s = (\frac{d}{dt} + \lambda)\tilde{x} = \dot{\tilde{x}} + \lambda\tilde{x} \rightsquigarrow \dot{s} = f + u - \ddot{x}_d + \lambda\dot{\tilde{x}}$$

- Best approximation $\hat{u}$ to achieve $\dot{s} = 0$
$$\hat{u} = -\hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}}$$

- The feedback control strategy is chosen intuitive "*if the error is negative, push hard enough in the positive direction (and conversely)*"

- To satisfy (4), a term discontinuous across the surface $s = 0$:
$$u = \hat{u} - ksgn(s)$$

where
$$\begin{array}{ll} sgn(s) = 1 & if\ \ s > 0 \\ sgn(s) = -1 & if\ \ s < 0 \end{array}$$

- Note that this strategy works only for first-order systems.
- By choosing k to be large enough (4) can be guaranteed

$$\frac{1}{2}\frac{d}{dt}s^2 = \dot{s}.s = (f - \hat{f})s - k|s|$$

,

- letting $k = F + \eta \rightsquigarrow \frac{1}{2}\frac{d}{dt}s^2 \leq -\eta|s|$
- **Integral Control**: To minimize the reaching time and make $s(t=0)=0$, one can use integral control, i.e. $\int_0^t \tilde{x}(r)dr$ as variable of interest.
  - The previous example is third order relative to this variable, so s: $s = (\frac{d}{dt} + \lambda)^2(\int_0^t \tilde{x}(r)dr) = \dot{\tilde{x}} + 2\lambda\tilde{x} + \lambda^2 \int_0^t \tilde{x}(r)dr$
  - The approximation of control law will be changed to

$$\hat{u} = -\hat{f} + \ddot{x}_d - 2\lambda\dot{\tilde{x}} - \lambda^2\tilde{x}$$

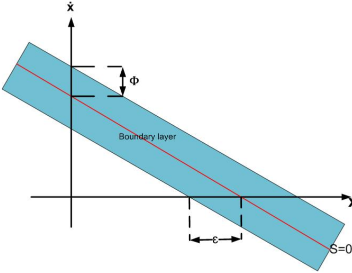
  - The control law, u and k will remain the same
  - Now if $\tilde{x}(0) \neq 0 \rightsquigarrow s = \dot{\tilde{x}} + 2\lambda\tilde{x} + \lambda^2 \int_0^t \tilde{x}(r)dr - \dot{\tilde{x}}(0) - 2\lambda\tilde{x}(0)$
  - $\therefore$ Although $\tilde{x}(0) \neq 0$, $s(t=0)=0$

Gain Margins

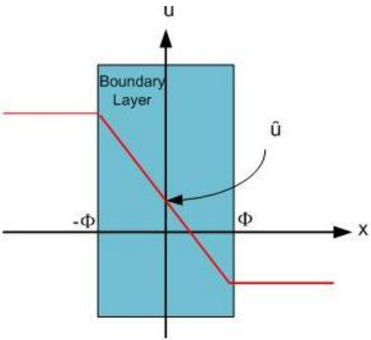
- Consider
$$\ddot{x} = f + bu$$
where the control gain , b which is may be time-varying or state-dependent is unknown, but of known bounds
$$0 < b_{min} \le b \le b_{max}$$
    - choose estimation of b as its geometric mean of bounds: $\hat{b} = (b_{min} b_{max})^{1/2}$.
    - $\therefore\ \beta^{-1} \le \frac{\hat{b}}{b} \le \beta$,
    - $\beta = (b_{max}/b_{min})^{1/2}$ is gain margin
- With s and $\hat{u}$ defined in previous example $u = \hat{b}^{-1}[\hat{u} - ksgn(s)]$
- $\dot{s} = (f - b\hat{b}^{-1}\hat{f}) + (1 - b\hat{b}^{-1})(-\ddot{x}_d + \lambda\dot{\tilde{x}}) - b\hat{b}^{-1}ksgn(s)$
- $\therefore$ to satisfy sliding condition
$k \ge |\hat{b}b^{-1}f - \hat{f} + (\hat{b}b^{-1} - 1)(-\ddot{x}_d + \lambda\dot{\tilde{x}})| + \eta\hat{b}b^{-1}$
- Since $f = \hat{f} + (f - \hat{f})$, where $|f - \hat{f}| \le F \rightsquigarrow k \ge \beta(F + \eta) + (\beta - 1)|\hat{u}|$

Continuous Approximations of Switching Control Laws

- For system dynamics (1) a unique smooth control to track a feasible trajectory is $u(t) = b(x_d)^{-1}[\ddot{x}_d - f(x_d)]$
- Control laws obtained by using sliding control which provides "perfect" tracking in the face of model uncertainty, are discontinuous across the surface $S(t)$, $\rightsquigarrow$ chattering.
- In general, chattering is undesirable, since it causes high control activity, and may excite high-frequency dynamics neglected in modeling
- The chattering is avoided by smoothing out the control discontinuity in a thin boundary layer neighboring the switching surface $B(t) = \{X, |s(x;t)| \leq \Phi\}, \quad \Phi > 0$ is the boundary layer thickness $\varepsilon = \Phi/\lambda^{n-1}$ is the boundary width



- Outside of $B(t)$, the control law u is like before to guarantee that the boundary layer is invariant
  - All trajectories starting inside $B(t=0)$ remain inside $B(t)$ for all $t>0$
- Inside $B(t)$, u is interpolated
  - For instance,, inside $B(t)$, in the expression of u replace $sgn(s)$ by s/Φ, as shown in Fig
- As it has been shown before, instead of perfect tracking, tracking to within a guaranteed precision $\varepsilon$is guaranteed.
  - For all trajectories starting inside $B(t=0)$ $\forall t \geq 0 |\tilde{x}^{(i)}| \leq (2\lambda)^i \varepsilon \;\; i = 1, ..., n-1$



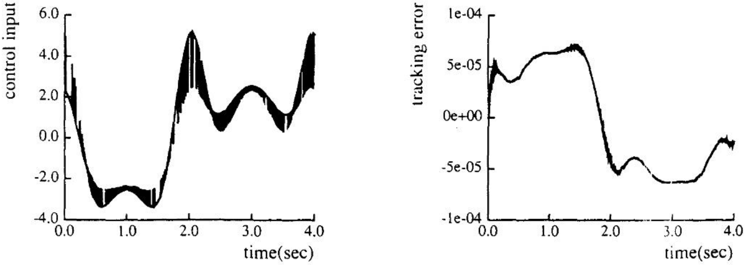
Example

- Consider the system dynamics

$$\ddot{x} + a(t)\dot{x}^2 \cos 3x = u$$

- $1 \leq a(t) \leq 2$, for simulation $a(t) = |\sin t| + 1$,
- $\lambda = 20, \ \eta = 0.1$
- $\hat{f} = 1.5\dot{x}^2 \cos 3x, \quad F = 0.5\dot{x}^2 |\cos 3x|$
- By using the switching control law: $u = \hat{u} - ksgn(s) =$ $1.5\dot{x}^2 cos3x + \ddot{x}_d - 20\dot{\tilde{x}} - (0.5\dot{x}^2|\cos 3x| + 0.1)sgn(\dot{\tilde{x}} + 20\tilde{x})$

Example Cont'd

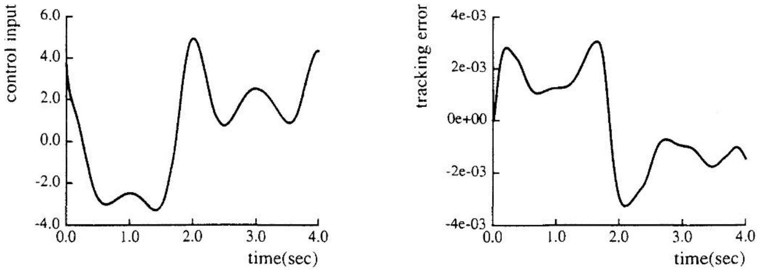


Switched control input and resulting tracking performance

- Tracking performance is excellent at the price of high control chattering

Example Cont'd

- Modify control law by considering a thin boundary layer of thickness 0.1
- $u = \hat{u} - k sat(s/\Phi) =$
  $1.5\dot{x}^2 cos 3x + \ddot{x}_d - 20\dot{\tilde{x}} - (0.5\dot{x}^2|\cos 3x| + 0.1) sat((\dot{\tilde{x}} + 20\tilde{x})/0.1)$



: Smooth control input and resulting tracking performance

- The tracking is not as perfect as before but acceptable, instead the control law is smooth

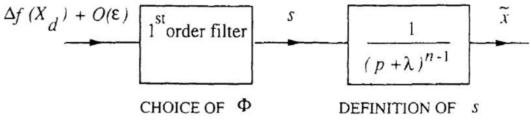
- The smoothing of control discontinuity inside B(t) actually assigns a low pass filter structure to the local dynamics of the variables to eliminating chattering
- Recognizing this filter-like structure allows us to
  - tune up the control law by selecting λ and Φ properly s.t achieve a trade-off between tracking precision and robustness to unmodeled dynamics.
- Φ can be made time varying
- Case 1: $b = \hat{b} = 1$
  - Φ is TV $\rightsquigarrow$ the sliding condition (4) to guarantee the decreasing distance to the boundary layer is changed to:

$$\|s\| \geq \Phi : \quad \frac{1}{2}\frac{d}{dt}s^2 \leq (\dot{\Phi} - \eta)|s| \tag{6}$$

    - The boundary layer attraction $\uparrow$ when the boundary layer $\downarrow$ ($\dot{\Phi} < 0$)
    - The boundary layer attraction $\downarrow$ when the boundary layer $\uparrow$ ($\dot{\Phi} > 0$)

Case 1: $b = \hat{b} = 1$

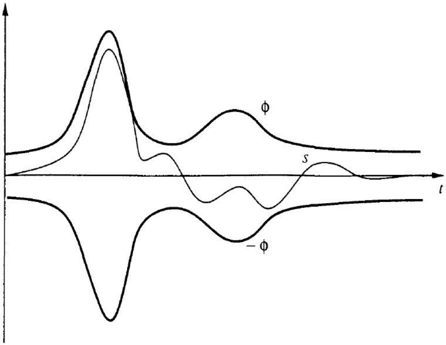
- the control signal is modified as:
  $u = \hat{u} - \bar{k}(x) sat(s/\Phi)$
  - $\bar{k}(x) = k(x) - \dot{\Phi}$
  - $sat(y) = \begin{cases} y & if\ |y| \leq 1 \\ sgn(y) & otherwise \end{cases}$
- So the system trajectories inside the boundary layer:
  $\dot{s} = -\bar{k}(x)\frac{s}{\Phi} - \Delta f(x) = -\bar{k}(x_d)\frac{s}{\Phi} + (-\Delta f(x_d) + O(\varepsilon))$
  where $\Delta f = \hat{f} - f$
- We can consider a first order filter:
  - its dynamic depends on desired state x_d
  - s : a measure of the algebraic distance to the surface $S(t)$ is its output
  - the "perturbations," (uncertainty $\Delta f(x_d)$) is its input



- s provides tracking error $\tilde{x}$ by further low pass filtering (2)
  - λ is break-frequency of the filter
  - It must be chosen to be "small" with respect to high-frequency unmodeled dynamics (such as unmodeled structural modes or neglected time delays)
- Let us define Φ based on bandwidth λ: $\frac{\bar{k}(x_d)}{\Phi} = \lambda$
- and:

$$\dot{\phi} + \lambda\Phi = k(x_d) \tag{7}$$
$$\bar{k}(x) = k(x) - k(x_d) + \lambda\Phi$$

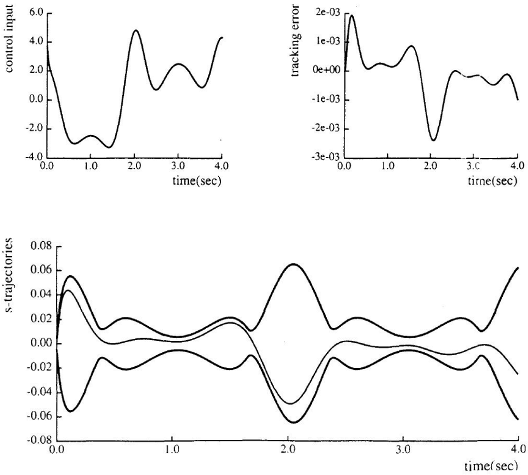
- The boundary layer thickness Φ is defined based on the evolution of dynamic model uncertainty
- Control signal depends on s
- s-trajectory represents a TV measure of the validity of the assumptions on model uncertainty
- tracking error $\tilde{x}$ is a filtered version of s



Example

- Recall the previous example and modify the control properly:
- $u = 1.5\dot{x}^2 cos 3x + \ddot{x}_d - 20\dot{\tilde{x}} - (0.5\dot{x}^2|\cos 3x| + \eta + \dot{\Phi}) sat((\dot{\tilde{x}} + 20\tilde{x})/\Phi)$
  $\dot{\Phi} = -\lambda\Phi + 0.5\dot{x_d}^2|\cos 3x_d| + \eta$
  - $\dot{x}_d(0) = 0,\ \eta = 0.1,\ \lambda = 20,\ \Phi(0) = \frac{\eta}{\lambda}$
- Max of the Φ is the same as the constant value of Φ in previous example
- The tracking error is about 4 times better

Example Cont'd



Case 2: $\beta \neq 1$

- Define: $\beta_d = \beta(x_d) = \frac{b(x_d)}{\hat{b}(x_d)}$
- If $k(x_d) \geq \frac{\lambda\Phi}{\beta_d} \Rightarrow \dot{\Phi} + \lambda\Phi = \beta_d k(x_d)$
- If $k(x_d) \leq \frac{\lambda\Phi}{\beta_d} \Rightarrow \dot{\Phi} + \frac{\lambda\Phi}{\beta_d^2} = \frac{k(x_d)}{\beta_d}$
- $\Phi(0) = \beta_d k(x_d(0))/\lambda$
- Modify $\lambda = \frac{\bar{k}(x_d)\beta_d}{\Phi}$
- And finally $\bar{k}(x) = k(x) - k(x_d) + \frac{\lambda\Phi}{\beta_d}$

Remarks

1. The desired trajectory x_d must be smooth enough not to excite the high-frequency unmodeled dynamics.
2. The sliding control guarantees the best tracking performance given the desired control bandwidth and the extent of parameter uncertainty.
3. If the model or its bounds are so imprecise that F can only be chosen as a large constant, then define Φ a large constant, s.t. the term $\bar{k} sat(s/\Phi) = \lambda s/\beta$ $\leadsto$like simple P.D.

Remarks

4. For exceptional disturbances which their intensity is high s.t may take the traj. out of the boundary:
    - If integral control is applied, the integral term in the control may become unreasonably large
    - once the disturbance stops, the system goes through large amplitude oscillations in order to return to the desired trajectory (integrator windup)
    - It is a potential cause of instability because of saturation effects and physical limits on the motion.
    - Solution: As long as the system is outside the boundary layer maintain the integral term constant
    - When the system remains in the boundary layer (returns to normal case after the exceptional disturbance) integration can resume

Example:

- Consider the following system

$$\begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 4 & -1.2 \end{bmatrix} \begin{bmatrix} \theta \\ \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ -0.2 \\ -20 \end{bmatrix} u$$
$$y = \theta - \alpha$$

- The transfer function will be: $\frac{y}{u} = 0.2\frac{(s+10.8)(s-9.8)}{s(s+3.1)(s-0.9)}$
- It is non minimum phase
- Taking one time derivative of output yields: $\dot{y} = -y + \theta + 0.2u$
- $\therefore$ The internal dynamics will be:

$$\begin{bmatrix} \dot{\theta} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -1.2 \end{bmatrix} \begin{bmatrix} \theta \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (-4y - 20u)$$

Example Cont'd

- If we consider $u = -5\theta$ the internal dynamics will be:

$$\begin{bmatrix} \dot{\theta} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 104 & -1.2 \end{bmatrix} \begin{bmatrix} \theta \\ q \end{bmatrix}$$

  - Eigne values: -10.8, 9.6
  - The system is unstable
- The sliding surface: $s = y = 0$
- Since the internal is not stable, no limited control signal can provide $y = 0$
- Consider $u = -sgn(y)$
- The results in the next slide confirm that the classical siding mode cannot control the non minimum phase systems

Example Cont'd