Computational Intelligence
Lecture 9: Fuzzy Control I

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Takagi-Sugeno-Kang Fuzzy System (TSK)[1]

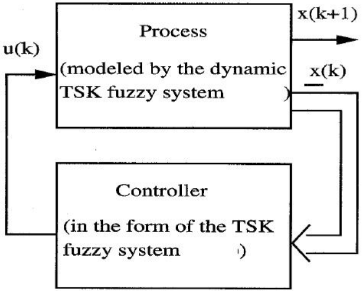
- A TSK fuzzy system is constructed from the following rules:
  IF x_1 is C_1^l and ... and x_n is C_n^l THEN $y^l = f(x_1, ..., x_n)$
- $y^l = f(x_1, ..., x_n)$ is a crisp function, and can be any general fcn.
- Usually two types of TSK fuzzy system is applied
  1. Zero-Order Sugeno Model
     - y^l is const.
     - It is a special case of the product inf. , singleton fuzzifier,
  2. First-Order Sugeno Model
     - y^l is a linear fcn. of x_i
- The output of the TSK fuzzy system is computed as the weighted average of the y^l's
  $y^* = \frac{\sum_{l=1}^{M} y^l w^l}{\sum_{l=1}^{M} w^l}$
  where $w^l = \prod_{i=1}^{n} \mu_{C_i^l}(x_i)$

Dynamic TSK Fuzzy System

- Output of a TSK fuzzy system appears as one of its inputs:
  IF $x(k)$ is A_1^p and ... and $x(k-n+1)$ is A_n^p and $u(k)$ is B^p THEN
  $x^p(k+1) = a_1^p x(k) + \ldots + a_n^p x(k-n+1) + b^p u(k)$
  - A^p and B^P are fuzzy sets
  - a^p and b^P are const., $p = 1, 2, ..., N$,
  - $u(k)$:input to the system
  - $\mathbf{x}(k) = (x(k), x(k-1), ..., x(k-n+l))^T \in R^n$: the state vector of the system.
- Output of the TSK is
  $x^*(k+1) = \frac{\sum_{p=1}^N x^p(k+1)v^p}{\sum_{p=1}^N v^p}$
  where $v^p = \prod_{i=1}^n \mu_{A_i^p}[x(k-i+1)]\mu_{B^p}[u(k)]$
- Dynamic TSK fuzzy system can be applied to model dynamics of a plant

Closed-Loop Dynamics of Fuzzy Model with Fuzzy Controller

- Consider a feedback control system
  - The process under control is modeled by the dynamic TSK fuzzy model
  - The controller is the TSK fuzzy system with $c_0^l = 0$ and $x_i = x(k-i+1)$ for $i = 1, 2, ..., n$



- The closed-loop fuzzy control system is equivalent to the dynamic TSK fuzzy system by the following rules:
  IF $x(k)$ is $(C_1^l \ and \ A_1^p)$ and and $x(k-n+1)$ is $(C_n^l \ and \ A_n^p)$ THEN $x^{lp}(k+1) = \sum_{i=1}^{n}(a_i^p + b^p c_i^l)x(k-i+1)$
  - $u(k)$: the output of the controller,
  - $l = 1, 2, ..., M, \ \ p = 1, 2, ..., N$
  - fuzzy sets $(C_i^l \ andA_i^p)$ are characterized by the mem. fcn. $\mu_{C_i^l}(x(k-i+l)), \mu_{A_i^p}(x(k-i+1))$.
- The output of this dynamic TSK fuzzy system:

  $$x(k+1) = \frac{\sum_{l=1}^{M}\sum_{p=1}^{N} x^{lp}(k+1)w^l v^p}{\sum_{l=1}^{M}\sum_{p=1}^{N} w^l v^p}$$

  where
  - $w^l = \prod_{i=1}^{n} \mu_{C_i^l}(x(k-i+1))$
  - $v^p = \prod_{i=1}^{n} \mu_{A_i^p}[x(k-i+1)]\mu_{B^p}[u(k)]$

Stability Analysis of the Dynamic TSK Fuzzy System

- Consider System dynamics $x(k+1) = Ax(k)$
- Based on Lyapunov theorem, this system is globally asymptotically stable iff $\exists P > 0$ s.t. $A^T PA - P < 0$

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- Now for the TSK dynamical model define:
  - $\mathbf{x}(k) = [x(k) \ldots x(k-n_1)]^T$
  - $A_p = \begin{bmatrix} a_1^p & a_2^p & \ldots & a_n^p \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & \ldots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$
  - $b^p = 0$ (consider no input $u(k)$ for the system)
  - $\therefore$ output of the systems: $x(k+1) = \frac{\sum_{p=1}^{N} A_p x(k) v^p}{\sum_{p=1}^{N} v^p}$
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- The TSK system modeled above is stable if $\exists$ a common matrix $P > 0$ s.t. $A_p^T PA_p - P < 0$ for all $p = 1, 2, ..., N$.

Design of Stable Fuzzy Controllers for the Fuzzy Model

1. Use The following closed-loop fuzzy control system as a dynamic TSK fuzzy system.
   IF $x(k)$ is $(C_1^l \textit{ and } A_1^p)$ and and $x(k-n+1)$ is $(C_n^l \textit{ and } A_n^p)$ THEN $x^{lp}(k+1) = \sum_{i=1}^{n}(a_i^p + b^p c_i^l)x(k-i+1)$
   - The output:
     $x(k+1) = \frac{\sum_{l=1}^{M}\sum_{p=1}^{N} x^{lp}(k+1) w^l v^p}{\sum_{l=1}^{M}\sum_{p=1}^{N} w^l v^p}$
     where $w^l = \prod_{i=1}^{n} \mu_{C_i^l}(x(k-i+1))$
     $v^p = \prod_{i=1}^{n} \mu_{A_i^p}[x(k-i+1)]\mu_{B^p}[u(k)]$
   - a_i^p and b^p, and $\mu_{A_i^p}$ are known
   - the controller parameters $c_i^p, \mu_{C_i^l}$ should be designed

2. Choose c_i^l and $A_{lp} = \begin{bmatrix} a_1^p + b^p c_l^l & a_2^p + b^p c_2^l & \dots & a_n^p + b^p c_n^l \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \dots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$ where $l = 1, ..., M; p = 1, ..., N$

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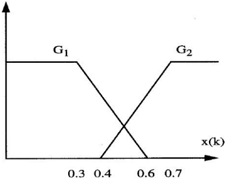
3. Find a common matrix $P > 0$ s.t. $A_{lp}^T P A_{lp} - P < 0$ for all $p = 1, 2, ..., N; l = 1, ..., M$. If you could not find such P, fo back to step 2 and redefine a new set of c_i^l's

Example

- Consider a TSK dynamical system:
  - IF $x(k)$ is G_1 THEN
    $x^1(k+1) = 2.18x(k) - 0.59x(k-1) - 0.603u(k)$
  - IF $x(k)$ is G_2 THEN
    $x^2(k+1) = 2.26x(k) - 0.36x(k-1) - 1.120u(k)$
- and TSK contoller:
  - IF $x(k)$ is G_1 THEN
    $u^1(k) = c_1^1 x(k) + c_2^1 x(k-1)$
  - IF $x(k)$ is G_2 THEN
    $u^2(k) = c_1^2 x(k) + c_2^2 x(k-1)$



Example Cont'd

- Step 1: design a closed loop TSK system
  - IF $x(k)$ is (G_1, G_1) THEN
    $x^{11}(k+1) = (2.18 - 0.603c_1^1)x(k) + (-0.59 - 0.603c_2^1)x(k-1)$
  - IF $x(k)$ is (G_1, G_2) THEN
    $x^{12}(k+1) = (2.18 - 0.603c_1^2)x(k) + (-0.59 - 0.603c_2^2)x(k-1)$
  - IF $x(k)$ is (G_2, G_1) THEN
    $x^{21}(k+1) = (2.26 - 1.120c_1^1)x(k) + (-0.63 - 1.120c_2^1)x(k-1)$
  - IF $x(k)$ is (G_2, G_2) THEN
    $x^{22}(k+1) = (2.26 - 1.120c_1^2)x(k) + (-0.63 - 1.120c_2^2)x(k-1)$

- Step 2: define matrices A_{lp}:
$A_{11} = \begin{bmatrix} 2.18 - 0.603c_1^1 & -0.59 - 0.603c_2^1 \\ 1 & 0 \end{bmatrix}; A_{12} =$
$\begin{bmatrix} 2.18 - 0.603c_1^2 & -0.59 - 0.603c_2^2 \\ 1 & 0 \end{bmatrix}; A_{21} =$
$\begin{bmatrix} 2.26 - 1.120c_1^1 & -0.63 - 1.120c_2^1 \\ 1 & 0 \end{bmatrix}; A_{22} =$
$\begin{bmatrix} 2.26 - 1.120c_1^2 & -63 - 1.120c_2^2 \\ 1 & 0 \end{bmatrix}$
- Step 3: by trail and error proper c_i^l are:
$c_1^1 = 1.564; c_2^1 = 0.223; c_1^2 = 0.912; c_2^2 = 0.079$

M. Sugeno, "An introductory survey of fuzzy control," *Inf. Sci* vol. 36 , pp. 59–83, 1985.