Signals and Systems
Lecture 7: Laplace Transform

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Introduction

- We had defined e^{st} as a basic function for CT LTI systems,s.t. $e^{st} \longrightarrow H(s)e^{st}$
- In Fourier transform $s = j\omega$
- In Laplace transform $s = \sigma + j\omega$
- By Laplace transform we can
  - Analyze wider range of systems comparing to Fourier Transform
  - Analyze both stable and unstable systems
- The bilateral Laplace Transform is defined:

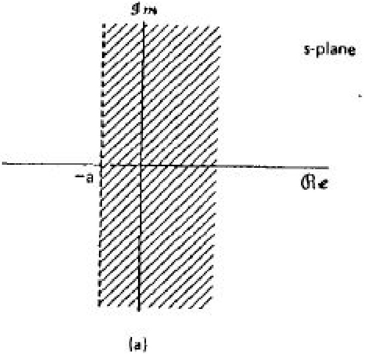
$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\ \Rightarrow X(\sigma + j\omega) &= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt \\ &= \mathcal{F}\{x(t)e^{-\sigma t}\} \end{aligned}$$

Region of Convergence (ROC)

- Note that: $X(s)$ exists only for a specific region of s which is called Region of Convergence (ROC)
- ROC: is the $s = \sigma + j\omega$ by which $x(t)e^{-\sigma}$ converges:
  $ROC : \{s = \sigma + j\omega \; s.t. \; \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty\}$
  - Roc does not depend on ω
  - Roc is absolute integrability condition of $x(t)e^{-\sigma t}$
- If $\sigma = 0$, i,e, $s = j\omega \rightsquigarrow X(s) = \mathcal{F}\{x(t)\}$
- ROC is shown in s-plane
- The coordinate axes are $\mathcal{R}e\{s\}$ along the horizontal axis and $\mathcal{I}m\{s\}$ along the vertical axis.

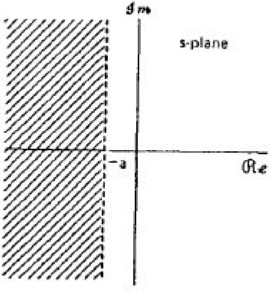
Example

- Consider $x(t) = e^{-at}u(t)$
- $X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \frac{-1}{s+a}e^{-(s+a)t}|_0^{\infty} = \frac{-1}{s+a}(e^{-(s+a)\infty} - 1)$
- If $\mathcal{R}e(s+a) > 0 \rightsquigarrow \mathcal{R}e(s) = \sigma > -\mathcal{R}e(a), X(s)$ is bounded
- $\therefore X(s) = \frac{1}{s+a}, \ ROC : \mathcal{R}e(s) > -\mathcal{R}e(a)$



Example

- Consider $x(t) = -e^{-at}u(-t)$
- $X(s) = -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}dt = \frac{1}{s+a}e^{-(s+a)t}|_{-\infty}^{0} = \frac{1}{s+a}(1 - e^{(s+a)\infty})$
- If $\mathcal{R}e(s+a) < 0 \rightsquigarrow \mathcal{R}e(s) = \sigma < -\mathcal{R}e(a), X(s)$ is bounded
- $\therefore X(s) = \frac{1}{s+a}, \; ROC : \mathcal{R}e(s) < -\mathcal{R}e(a)$



- In the recent two examples two different signals had similar Laplace transform but with different Roc
- To obtain unique $x(t)$ both $X(s)$ and ROC is required
- If $x(t)$ is defined as a linear combination of exponential functions, $\rightsquigarrow$ its Laplace transform ($X(s)$) is rational
- In LTI expressed in terms of linear constant-coefficient differential equations, Laplace Transform of its impulse response (its transfer function) is rational
- $X(s) = \frac{N(s)}{D(s)}$
  - Roots of $N(s)$ zeros of X(s); They make X(s) equal to zero.
  - Roots of $D(s)$ poles of X(s); They make X(s) to be unbounded.

- To study the stability of LTI systems zeros and poles are illustrated in s-plane (pole-zero plot)
- number of poles and zeros are equal for $-\infty$ to ∞
  - Consider degree of $D(s)$ (# of poles): m; degree of $N(s)$ (# of zeros): n
  - If $m < n \rightsquigarrow$ There are $n - m = k$ poles in ∞
  - If $m > n \rightsquigarrow$ There are $m - n = k$ zeros in ∞

ROC Properties

- ROC only depends on σ
  - In s-plane Roc is strips parallel to $j\omega$ axis
- If $X(s)$ is rational, Roc does not contain any pole
  - Since $D(s) = 0$, makes $X(s)$ unbounded
- If $x(t)$ is finite duration and is absolutely integrable, then ROC is entire s-plane
- If $x(t)$ is right sided and $\mathcal{R}e\{s\} = \sigma_0 \in$ ROC then $\forall s \;\; \mathcal{R}e\{s\} > \sigma_0 \in$ ROC
- If $x(t)$ is left sided and $\mathcal{R}e\{s\} = \sigma_0 \in$ ROC then $\forall s \;\; \mathcal{R}e\{s\} \leq \sigma_0 \in$ ROC
- If $x(t)$ is two sided and $\mathcal{R}e\{s\} = \sigma_0 \in$ ROC then ROC is a strip in s-plane including $\mathcal{R}e\{s\} = \sigma_0$

ROC Properties

- If $X(s)$ is rational
  - the ROC is bounded between poles or extends to infinity,
  - no poles of $X(s)$ are contained in ROC
  - If $x(t)$ is right sided, then ROC is in the right of the rightmost pole
  - If $x(t)$ is left sided, then ROC is in the left of the leftmost pole
- If ROC includes $j\omega$ axis then $x(t)$ has FT

Inverse of Laplace Transform (LT)

- By considering σ fixed, inverse of LT can be obtained from inverse of FT:
- $x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\underbrace{\sigma + j\omega}_{s})e^{j\omega t}d\omega$
- $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma+j\omega)t}d\omega$
- assuming σ is fixed $\rightsquigarrow ds = jd\omega$
- $\therefore x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)e^{st}ds$
- If $X(s)$ is rational , we can use expanding the rational algebraic into a linear combination of lower order terms and then one may use
  - $X(s) = \frac{1}{s+a} \rightsquigarrow x(t) = -e^{-at}u(-t)$ if $\mathcal{R}e\{s\} < -a$
  - $X(s) = \frac{1}{s+a} \rightsquigarrow x(t) = e^{-at}u(t)$ if $\mathcal{R}e\{s\} > -a$
- Do not forget to consider ROC in obtaining inverse of LT!

LT Properties

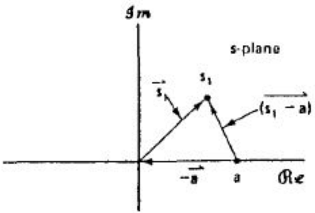
- Linearity: $ax_1(t) + bx_2(t) \Leftrightarrow aX_1(s) + bX_2(s)$
  - ROC contains: $R_1 \bigcap R_2$
  - If $R_1 \bigcap R_2 = \emptyset$ it means that LT does not exit
  - By zeros and poles cancelation ROC can be larger than $R_1 \bigcap R_2$
- Time Shifting: $x(t - T) \Leftrightarrow e^{-sT} X(s)$ with ROC$=R$
- Shifting in S-Domain: $e^{s_0 t} x(t) \Leftrightarrow X(s - s_0)$ with ROC$= R + \mathcal{R}e\{s_0\}$
- Time Scaling: $x(at) \Leftrightarrow \frac{1}{|a|} X(\frac{s}{a})$ with ROC $= \frac{R}{a}$
- Differentiation in Time-Domain: $\frac{dx(t)}{dt} \Leftrightarrow sX(s)$ with ROC containing R
- Differentiation in the s-Domain: $-tx(t) \Leftrightarrow \frac{dX(s)}{ds}$ with ROC $= R$
- Convolution: $x_1(t) * x_2(t) \Leftrightarrow X_1(s)X_2(s)$ with ROC containing $R_1 \cap R_2$

Analyzing LTI Systems with LT

- LT of impulse response is $H(s)$ which is named transfer function or system function.
- Transfer fcn can represent many properties of the system:
  - Causality: $h(t) = 0$ for $t < 0 \rightsquigarrow$ It is right sided
    - ROC of a causal system is a right-half plane
    - Note that the converse is not always correct
    - Example: $H(s) = \frac{e^s}{s+1}$, $\mathcal{R}e\{s\} > -1 \rightsquigarrow h(t) = e^{-(t+1)}u(t+1)$ it is none zero for $-1 < t < 0$
    - For a system with rational transfer fcn, causality is equivalent to ROC being the right-half plane to the right of the rightmost pole
  - Stability: $h(t)$ should be absolute integrable $\rightsquigarrow$ its FT converges
    - An LTI system is stable iff its ROC includes $j\omega$ axis ($0 \in$ ROC)
  - A causal system with rational $H(s)$ is stable iff all the poles of $H(s)$ have negative real-parts (are in left-half plane)

Geometric Evaluation of FT by Zero/Poles Plot

- Consider $X_1(s) = s - a$



  - $|X_1|$: length of X_1
  - $\measuredangle X_1$: angel of X_1
- Now consider $X_2(s) = \frac{1}{s-a} = \frac{1}{X_1(s)}$
  - $log X_2 = -log X_1$
  - $\measuredangle X_2 = -\measuredangle X_1$

- For higher order fcns:

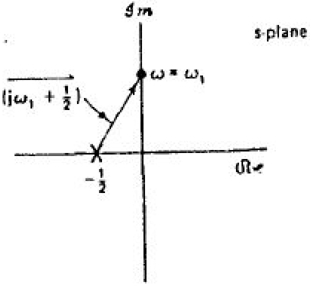
  $X(s) = M\frac{\prod_{i=1}^{R}(s-\beta_i)}{\prod_{j=1}^{P}(s-\alpha_j)}$

  - $|X(s)| = |M|\frac{\prod_{i=1}^{R}|s-\beta_i|}{\prod_{j=1}^{P}|s-\alpha_j|}$
  - $\measuredangle X(s) = \measuredangle M + \sum_{i=1}^{R}\measuredangle(s-\beta_i) - \sum_{j=1}^{R}\measuredangle(s-\alpha_j)$

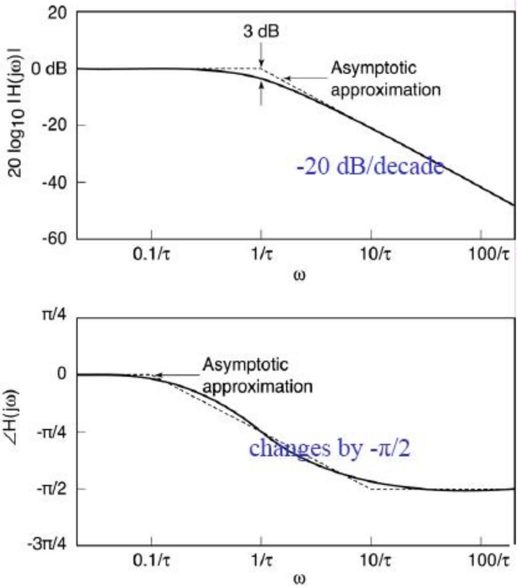
- Example:

  $H(s) = \frac{1/2}{s+1/2}, \quad \mathcal{R}e\{s\} > \frac{-1}{2}$

  - $h(t) = \frac{1}{2}e^{-t/2}u(t)$
  - $s(t) = [1 - e^{-t/2}]u(t)$
  - $H(j\omega) = \frac{1/2}{j\omega+1/2}$
  - $|H(j\omega)|^2 = \frac{(1/2)^2}{w^2+(1/2)^2}$
  - $\measuredangle H(j\omega) = -\tan^{-1}2\omega$
  - $0 < \omega < \infty \rightsquigarrow -\pi/2 < \measuredangle H(j\omega) < 0$
  - $\omega\uparrow \rightsquigarrow |H|\downarrow, \measuredangle H(j\omega)\downarrow$



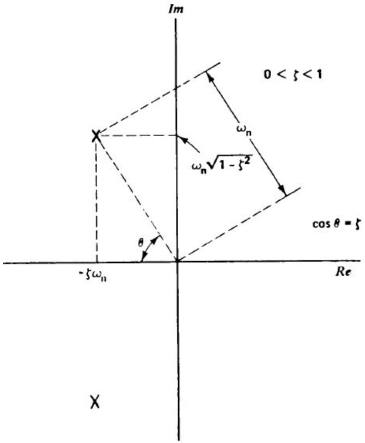
- Now let us substitute 2 with τ in the previous example
- $H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau}$
- $|H(j\omega)|^2 = \frac{(1/\tau)^2}{w^2 + (1/\tau)^2}, |H(j\omega)| = \begin{cases} 1 & \omega = 0 \\ \frac{1}{\sqrt{2}} & \omega = \frac{1}{\tau} \\ \frac{1}{\tau\omega} & \omega \gg \frac{1}{\tau} \end{cases}$
- $\measuredangle H(j\omega) = -\tan^{-1}\tau\omega = \begin{cases} 0 & \omega = 0 \\ \frac{-\pi}{4} & \omega = \frac{1}{\tau} \\ \frac{-\pi}{2} & \omega \gg \frac{1}{\tau} \end{cases}$
- Relation between real part of poles and response of the systems
  - τ is time constant of first order systems which control response speed of the systems
  - Poles are located at $-\frac{1}{\tau}$
  - The farther the poles from $j\omega$ axis $\rightsquigarrow$ cut-off freq. $\uparrow$, $\tau \downarrow$, the faster decaying the impulse response, the faster rise time of step response

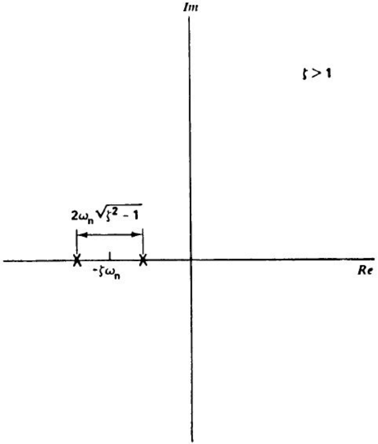

20
0 dB
-20
-40
-60
20 log10 |H(jω)|
3 dB
Asymptotic approximation
-20 dB/decade
0.1/τ
1/τ
10/τ
100/τ
ω
π/4
0
-π/4
-π/2
-3π/4
∠H(jω)
Asymptotic approximation
changes by -π/2
0.1/τ
1/τ
10/τ
100/τ
ω

Response for Second Order system

- $h(t) = M(e^{c_1 t} - e^{c_2 t})u(t)$
- $H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s-c_1)(s-c_2)}$
- $c_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
- $0 < \zeta < 1$: under damp (two complex poles), $c_2 = c_1^*$
- $\zeta = 1$ critically damp ($s = -\omega_n$)
- $\zeta > 1$: Over damp (two negative real poles)
- For fixed ω_n, $\zeta \uparrow\uparrow \rightsquigarrow$, settling time for step response $\uparrow$

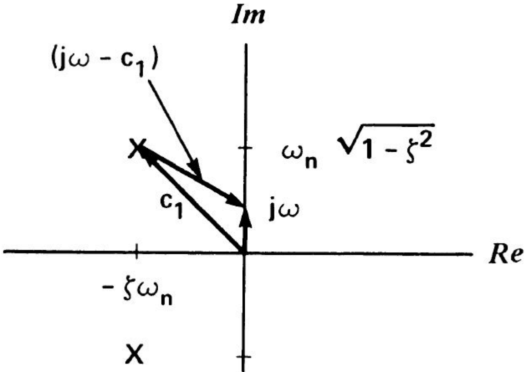
Zero-Pole Pattern of Second Order System



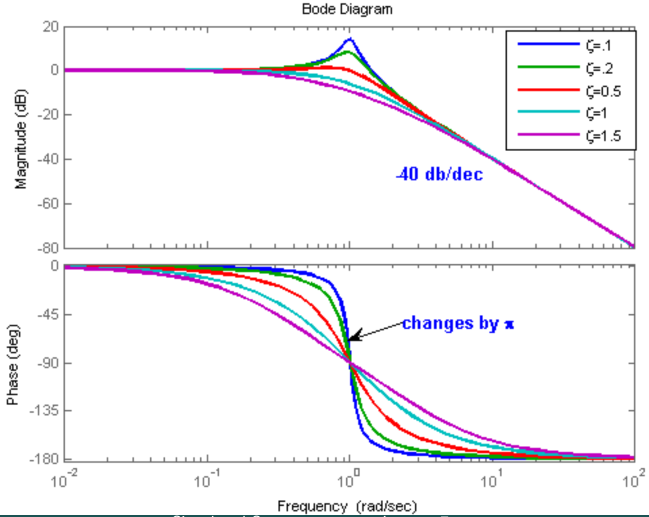


Freq. Response of Second Order System

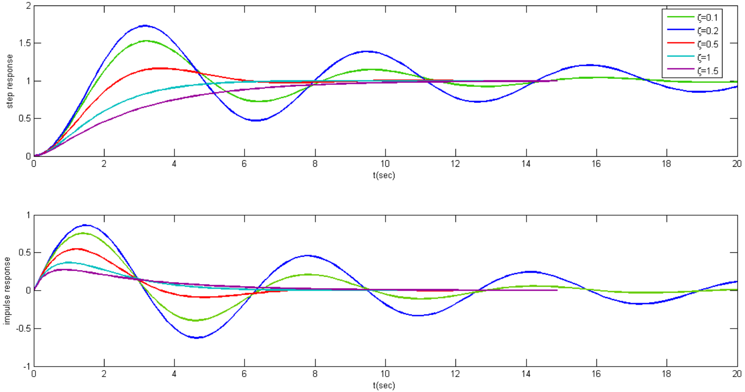
- $H(s) = \frac{\omega_n^2}{(s-c_1)(s-c_1^*)}$
- $H(j\omega) = H(s)|_{s=j\omega} = \frac{\omega_n^2}{(j\omega-c_1)(j\omega-c_1^*)}$



Bode Plot of $H(j\omega)$

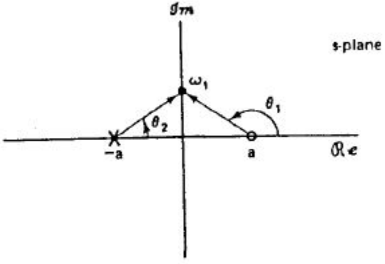


Impulse and Step Response of the second order system

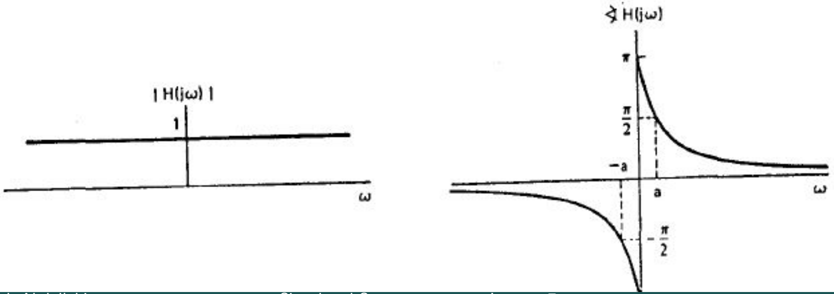


All Pass Filters

- Passes the signal in all freqs. with a little decreasing/increasing the magnitude
- Why do we use all-pass filters?
- $H(s) = \frac{s-a}{s+a} \quad \mathcal{R}e\{s\} > -a, \;\; a > 0$
- $|H(\jmath\omega)| = 1$
- $\measuredangle H(j\omega) = \theta_1 - \theta_2 = \pi - 2\theta_2 = \pi - 2tan^{-1}(\frac{\omega}{a}) = \begin{cases} \pi & \omega = 0 \\ \frac{\pi}{2} & \omega = a \\ 0 & \omega \gg a \end{cases}$


$\mathcal{I}m$
s-plane
ω_1
θ_1
θ_2
$-a$
a
$\mathcal{R}e$

(a)


$|H(j\omega)|$
1
ω
$\sphericalangle H(j\omega)$
π
$\frac{\pi}{2}$
$-a$
a
ω
$-\frac{\pi}{2}$

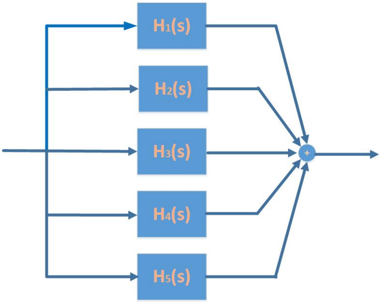
LTI Systems Description

- $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$
- $\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$
- $H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$
- ROC depends on
  - placement of poles
  - boundary conditions (right sided, left sided, two sided,...)

- High Order Systems can be expressed by connected simple order systems:
- Cascade Connection:

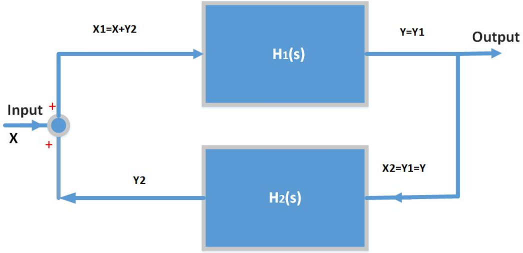
$$H(s) = H_1(s)H_2(s)H_3(s)H_4(s)H_5(s)$$

- Parallel Connection:



$$H(s) = H_1(s) + H_2(s) + H_3(s) + H_4(s) + H_5(s)$$

- Feedback Interconnection of two LTI systems:
  - $Y(s) = Y_1(s) = X_2(s)$
  - $X_1(s) = X(s) + Y_2(s) = X(s) + H_2(s)Y(s)$
  - $Y(s) = H_1(s)X_1(s) = H_1(s)[X(s) + H_2(s)Y(s)]$
  - $\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1-H_2(s)H_1(s)}$
  - ROC: is determined based on roots of $1 - H_2(s)H_1(s)$



Block Diagram Representation for Causal LTI Systems

- We can represent a transfer fcn by different methods:
- Example: $H(s) = \frac{2s^2+4s-6}{s^2+3s+2}$
  1. $H(s) = (2s^2 + 4s - 6)\frac{1}{s^2+3s+2}$
  2. Assuming it is causal so it is at initial rest
     - $W(s) = \frac{1}{s^2+3s+2}X(s) \Leftrightarrow \frac{d^2w}{dt^2} + 3\frac{dw}{dt} + 2w = x(t)$
     - $Y(s) = (2s^2 + 4s - 6)W(s) \Leftrightarrow y(t) = 2\frac{dw^2}{dt^2} + 4\frac{dw}{dt} - 6w$
  3. $H(s) = 2 + \frac{6}{s+2} - \frac{8}{s+1}$
  4. $H(s) = \frac{2(s-1)}{s+2}\frac{s+3}{s+1}$

Stability Analysis by Routh-Hurwitz

- Remind: A system with rational transfer fcn is causal and stable if all of its poles are in LHP.
- $H(s) = \frac{N(s)}{D(s)}$, $D(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$
- How can we verify the stability of this system?
  - Method 1: Find the roots of $D(s)$
    - If n is large, it is difficult to find: –(
  - Method 2: Routh-Hurwitz method
    - Provide the following table

$$
\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\
\vdots & \vdots & & & \\
s^0 & h_{n-1} & & &
\end{array}
$$

    - First row includes odd coefficients of $D(s)$
    - Second row includes even coefficients of $D(s)$

Stability Analysis by Routh-Hurwitz

- b_i, c_i are defined as follows:

$$b_{n-1} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}, \quad b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_{n-1} = -\frac{1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}, \quad c_{n-3} = -\frac{1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}$$

- Follow the same rule for other rows parameters
- # of RHP root of $D(s)$ equals to # of signs changing in the first column of the table
- Necessary condition for using Routh-Horwitz method is that all coefficients of $D(s)$ should exist and have similar sign(otherwise there are more than one pole on imaginary axis, it is not stable)
- Necessary and Sufficient conditions for stability is that no signs changing appears in the first column of the Routh-Horwitz table

- Initial Value Theorem: If $x(t) = 0$ for $t < 0$ and $x(t)$ does not contain any impulse or higher order singularities at the origin then $x(0^+) = \lim_{s\to\infty} sX(s)$
  - $X(s)$ may include a simple pole at the origin which represents a step signal.
  - More than one pole at the origin and in $j\omega$ axis make the signal oscillating
- Final Value Theorem: If $x(t) = 0$ for $t < 0$ and $x(t)$ is bounded when $t \longrightarrow \infty$ then $x(\infty) = \lim_{s\to 0} sX(s)$
- Consider $H(s) = \frac{N(s)}{D(s)}$, n is degree of $N(s)$, d is degree of $D(s)$:
  - $H(0^+) = \begin{cases} 0 & d > n+1 \\ \text{constant value} \neq 0 & d = n+1 \\ \infty & d < n+1 \end{cases}$

Unilateral LT

- It is used to describe causal systems with nonzero initial conditions: $\mathcal{X}(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt = \mathcal{UL}\{x(t)\}$
- If $x(t) = 0$ for $t < 0$ then $\mathcal{X}(s) = X(s)$
- Unilateral LT of $x(t)$ = Bilateral LT of $x(t)u(t^-)$
- If $h(t)$ is impulse response of a causal LTI system then $H(s) = \mathcal{H}(s)$
- ROC is not necessary to be recognized for unilateral LT since it is always a right-half plane
- For rational $\mathcal{X}(s)$, ROC is in right of the rightmost pole

Similar Properties of Unilateral and Bilateral LT

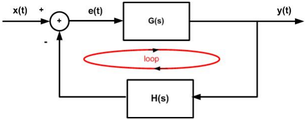
- Convolution: Note that for unilateral LT, If both $x_1(t)$ and $x_2(t)$ are zero for $t < 0$, then $\mathcal{X}(s) = \mathcal{X}_1(s)\mathcal{X}_2(s)$
- Time Scaling
- Shifting in s domain
- Initial and Finite Theorems: they are indeed defined for causal signals
- Integrating: $\int_{0^-}^{t} x(\tau)d\tau = x(t) * u(t) \overset{UL}{\Leftrightarrow} \mathcal{X}(s)\mathcal{U}(s) = \frac{1}{s}\mathcal{X}(s)$
- The main difference between $\mathcal{UL}$ and LT is in time differentiation:
  - $\mathcal{UL}\{\frac{dx(t)}{dt}\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$
  - Use the rule $\int fdg = fg - \int gdf$
  - $\therefore \mathcal{UL}\{\frac{dx(t)}{dt}\} = s\int_{0^-}^{\infty} x(t)e^{-st}dt + x(t)e^{-st}|_{0^-}^{\infty} = s\mathcal{X}(s) - x(0^-)$
  - $\mathcal{UL}\{\frac{dx(t)}{dt}\} = s\mathcal{X}(s) - x(0^-)$
  - $\mathcal{UL}\{\frac{d^2x(t)}{dt^2}\} = \mathcal{UL}\{\frac{d}{dt}\{\frac{dx(t)}{dt}\}\} = s(s\mathcal{X}(s) - x(0^-)) - \acute{x}(0^-) = s^2\mathcal{X}(s) - sx(0^-) - \acute{x}(0^-)$
  - Follow the same rule for higher derivatives

Example

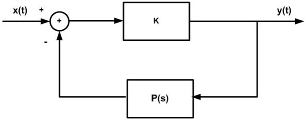
- Consider $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = x(t)$, where
  $y(0^-) = \beta = 3,\ \acute{y}(0^-) = \gamma = -5,\ x(t) = \alpha u(t) = 2u(t)$
- Take $\mathcal{UL}$:
  - $s^2\mathcal{Y}(s) - \beta\mathcal{Y}(s) - \gamma + 3(s\mathcal{Y}(s) - \beta) + 2\mathcal{Y}(s) = \mathcal{X}(s)$
  - $\mathcal{Y}(s) = \underbrace{\frac{\beta(s+3)+\gamma}{s^2+3s+2}}_{ZIR} + \underbrace{\frac{\mathcal{X}(s)}{s^2+3s+2}}_{ZSR}$
  - Zero State Response (ZSR): is a response in absence of initial values
    - $\mathcal{H}(s) = \frac{\mathcal{Y}(s)}{\mathcal{X}(s)}$
    - Transfer fcn is ZSR
    - ZSR: $\mathcal{Y}_1(s) = \frac{\alpha}{s(s+1)(s+2)} = \frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}$
    - $y_1(t) = (1 - 2e^{-t} + e^{-2t})u(t)$
  - Zero Input Response (ZIR): is a response in absence of input $(x(t) = 0)$
  - ZIR: $\mathcal{Y}_2(s) = \frac{3(s+3)-5}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{2}{s+2}$
  - $y_2(t) = (e^{-t} + 2e^{-2t})u(t)$
- $y(t) = y_1(t) + y_2(t)$

Feed Back Applications

- Closed loop Transfer fcn: $Q(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{Open\ \ loop\ \ Gain}{1-Loop\ \ Gain}$

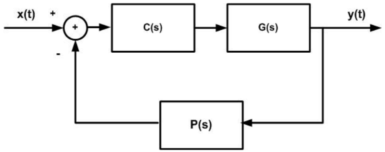


1. Inverting



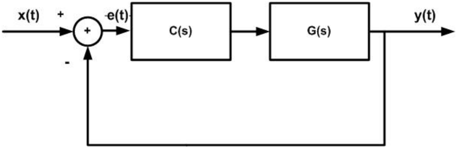
- $Q(s) = \frac{K}{1+Kp(s)}$
- If choose K s.t. $Kp(s) \gg 1$ then $Q(s) \simeq \frac{1}{p(s)}$
- Example: For a capacitor, consider i as output and v as input, it is a differentiator
- By using the above interconnection, we can make an integrator

2. Stabilizing Unstable Systems



- $G(s)$ is unstable
- We should define $P(s)$ and $C(s)$ to make closed-loop system stable (poles of closed-loop system be in LHP)
- $Q(s) = \frac{C(s)G(s)}{1+C(s)P(s)G(s)}$
- Example 1: $G(s) = \frac{1}{s-2}, \;\; C(s) = K, \;\; P(s) = 1$
- $Q(s) = \frac{K}{s-2+K}$
- Choosing $K > 2$ make it stable
- Example 2: $G(s) = \frac{1}{s^2-4}$
- By $C(s) = K$ cannot be stabilized
- Choose $C(s) = K_1 + K_2 s$, $K_2 > 0$, and $K_1 > 4$ can stabilize the closed-loop system

3. Tracking



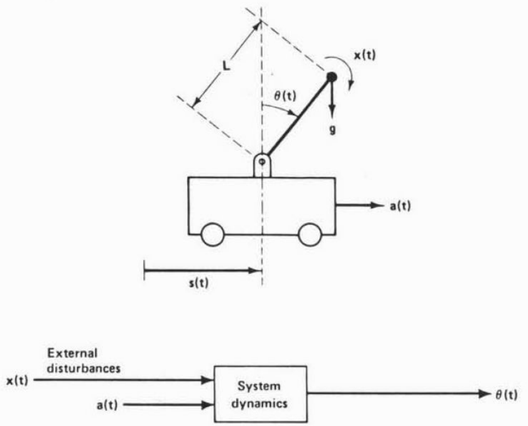
- Objective: Defining $C(s)$ s.t. $e(t) = x(t) - y(t) \to 0$ as $t \to \infty$
- $E(s) = \frac{1}{1+C(s)G(s)} X(s)$
- Consider $x(t)$ as unite step
- $\lim e(t)_{t\to\infty} = \lim sE(s)_{s\to 0} = \lim_{s\to 0} \frac{s}{1+C(s)G(s)} \frac{1}{s}$
- If we choose $C(s)$ s.t. $C(s)G(s) \gg 1$ then $e(t) \to 0$ as $t \to \infty$

4. Decreasing effect of disturbance

5. Decreasing Sensitivity to uncertainties

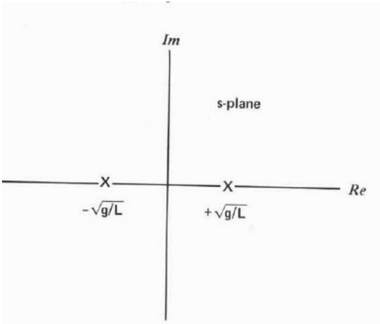
Inverted Pendulum

- Objective:
  Find proper $a(t)$ to make $\theta(t) = 0$



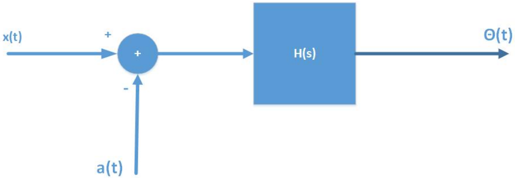
Inverted Pendulum

- System Dynamics:
  $L\frac{d^2\theta(t)}{dt^2} = g\sin[\theta(t)] + Lx(t) - a(t)\cos(\theta(t))$
- Linearize it: assuming $\theta(t)$ is small
  - $\sin(\theta(t)) = \theta(t)$
  - $\cos(\theta(t)) = 1$
  - $L\frac{d^2\theta(t)}{dt^2} = g\theta(t) + Lx(t) - a(t)$
  - LT:
    $\Theta(s) = \underbrace{\frac{1}{Ls^2 - g}}_{H(s)}[LX(s) - A(s)]$



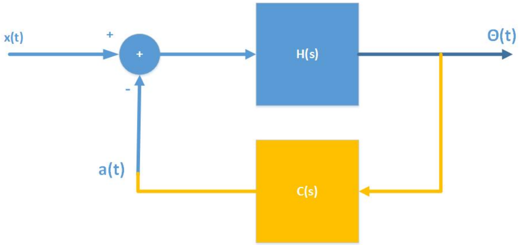
Inverted Pendulum

- $\Theta(s) = H(s)[LX(s) - A(s)]$
- $H(s) = \frac{1}{Ls^2 - g}$



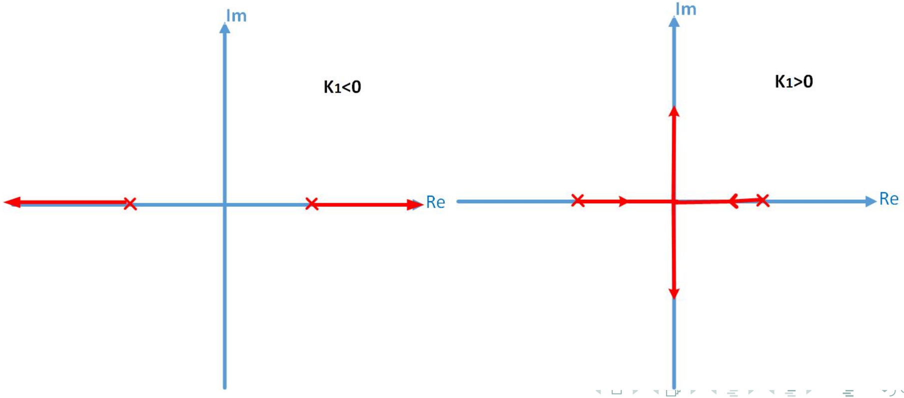
Inverted Pendulum

- $\Theta(s) = H(s)[LX(s) - A(s)]$
- $H(s) = \frac{1}{Ls^2 - g}$
- Using feedback connection, let us design a controller, $C(s)$ to make the pendulum in vertical position
- $\Theta(s) = \frac{LH(s)}{1 + C(s)H(s)} X(s)$



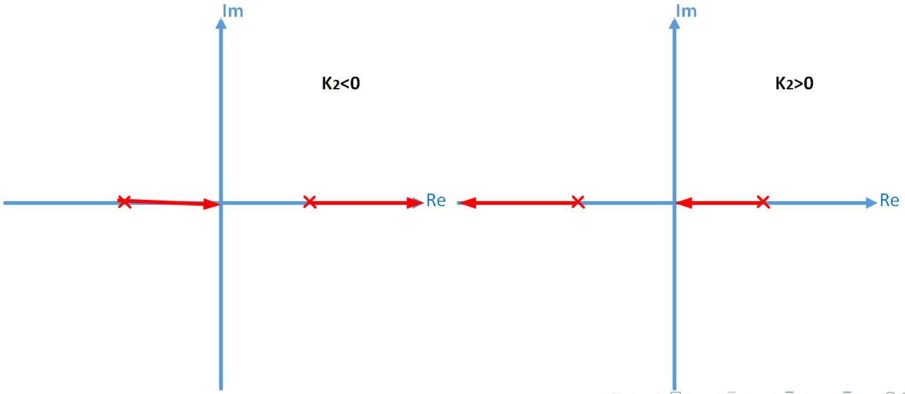
Proportional Feedback: $C(s) = K_1$

- $\Theta(s) = \frac{1}{s^2 - \frac{g - K_1}{L}} X(s)$
- Poles $s = \pm\sqrt{\frac{g - K_1}{L}}$



Derivative Feedback: $C(s) = K_2 s$

- $\Theta(s) = \frac{1}{s^2+s(K_2/L)-g/L} X(s)$
- Poles: $s = -\frac{K_2}{2L} \pm \sqrt{(\frac{K_2}{2L})^2 + \frac{g}{L}}$



Proportional+ Derivative (PD) Feedback: $C(s) = K_1 + K_2 s$

- $\Theta(s) = \frac{1}{s^2+s(K_2/L)-g/L+K_1/L} X(s)$
- Poles: $s = -\frac{K_2}{2L} \pm \sqrt{(\frac{K_2}{2L})^2 - \frac{K_1-g}{L}}$