

Signals and Systems

Lecture 7: Laplace Transform

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Introduction

- ROC Properties

- Inverse of LT

- LT Properties

Analyzing LTI Systems with LT

Geometric Evaluation

- All Pass Filters

LTI Systems Description

- Stability Analysis by Routh-Hurwitz

Unilateral LT

Feed Back Applications

Introduction

- ▶ We had defined e^{st} as a basic function for CT LTI systems, s.t.
 $e^{st} \rightarrow H(s)e^{st}$
- ▶ In Fourier transform $s = j\omega$
- ▶ In Laplace transform $s = \sigma + j\omega$
- ▶ By Laplace transform we can
 - ▶ Analyze wider range of systems comparing to Fourier Transform
 - ▶ Analyze both stable and unstable systems
- ▶ The **bilateral Laplace Transform** is defined:

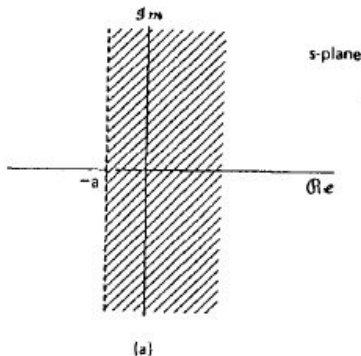
$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st} dt \\ \Rightarrow X(\sigma + j\omega) &= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t} dt \\ &= \mathcal{F}\{x(t)e^{-\sigma t}\} \end{aligned}$$

Region of Convergence (ROC)

- ▶ **Note that:** $X(s)$ exists only for a specific region of s which is called Region of Convergence (ROC)
- ▶ **ROC:** is the $s = \sigma + j\omega$ by which $x(t)e^{-\sigma t}$ converges:
 $ROC : \{s = \sigma + j\omega \text{ s.t. } \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty\}$
 - ▶ Roc does not depend on ω
 - ▶ Roc is absolute integrability condition of $x(t)e^{-\sigma t}$
- ▶ If $\sigma = 0$, i.e, $s = j\omega \rightsquigarrow X(s) = \mathcal{F}\{x(t)\}$
- ▶ ROC is shown in s-plane
- ▶ The coordinate axes are $\mathcal{R}e\{s\}$ along the horizontal axis and $\mathcal{I}m\{s\}$ along the vertical axis.

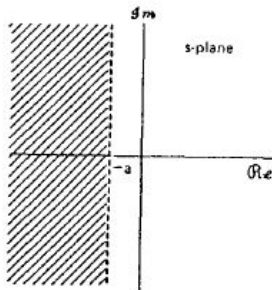
Example

- ▶ Consider $x(t) = e^{-at}u(t)$
- ▶ $X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \frac{-1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{-1}{s+a}(e^{-(s+a)\infty} - 1)$
- ▶ If $\text{Re}(s+a) > 0 \rightsquigarrow \text{Re}(s) = \sigma > -\text{Re}(a)$, $X(s)$ is bounded
- ▶ $\therefore X(s) = \frac{1}{s+a}$, ROC : $\text{Re}(s) > -\text{Re}(a)$



Example

- ▶ Consider $x(t) = -e^{-at}u(-t)$
- ▶ $X(s) = -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}dt = \frac{1}{s+a}e^{-(s+a)t}\Big|_{-\infty}^0 = \frac{1}{s+a}(1 - e^{(s+a)\infty})$
- ▶ If $\text{Re}(s+a) < 0 \rightsquigarrow \text{Re}(s) = \sigma < -\text{Re}(a)$, $X(s)$ is bounded
- ▶ $\therefore X(s) = \frac{1}{s+a}$, ROC : $\text{Re}(s) < -\text{Re}(a)$



- ▶ In the recent two examples two different signals had similar Laplace transform but with different Roc
- ▶ To obtain unique $x(t)$ both $X(s)$ and ROC is required
- ▶ If $x(t)$ is defined as a linear combination of exponential functions, \rightsquigarrow its Laplace transform ($X(s)$) is rational
- ▶ In LTI expressed in terms of linear constant-coefficient differential equations, Laplace Transform of its impulse response (its transfer function) is rational
- ▶ $X(s) = \frac{N(s)}{D(s)}$
 - ▶ Roots of $N(s)$ zeros of $X(s)$; They make $X(s)$ equal to zero.
 - ▶ Roots of $D(s)$ poles of $X(s)$; They make $X(s)$ to be unbounded.

- ▶ To study the stability of LTI systems zeros and poles are illustrated in s-plane (**pole-zero plot**)
- ▶ number of poles and zeros are equal for $-\infty$ to ∞
 - ▶ Consider degree of $D(s)$ (# of poles): m ; degree of $N(s)$ (# of zeros): n
 - ▶ If $m < n \rightsquigarrow$ There are $n - m = k$ poles in ∞
 - ▶ If $m > n \rightsquigarrow$ There are $m - n = k$ zeros in ∞

ROC Properties

- ▶ ROC only depends on σ
 - ▶ In s -plane Roc is strips parallel to $j\omega$ axis
- ▶ If $X(s)$ is rational, Roc does not contain any pole
 - ▶ Since $D(s) = 0$, makes $X(s)$ unbounded
- ▶ If $x(t)$ is finite duration and is absolutely integrable, then ROC is entire s -plane
- ▶ If $x(t)$ is right sided and $\mathcal{Re}\{s\} = \sigma_0 \in \text{ROC}$ then $\forall s \ \mathcal{Re}\{s\} > \sigma_0 \in \text{ROC}$
- ▶ If $x(t)$ is left sided and $\mathcal{Re}\{s\} = \sigma_0 \in \text{ROC}$ then $\forall s \ \mathcal{Re}\{s\} \leq \sigma_0 \in \text{ROC}$
- ▶ If $x(t)$ is two sided and $\mathcal{Re}\{s\} = \sigma_0 \in \text{ROC}$ then ROC is a strip in s -plane including $\mathcal{Re}\{s\} = \sigma_0$

ROC Properties

- ▶ If $X(s)$ is rational
 - ▶ the ROC is bounded between poles or extends to infinity,
 - ▶ no poles of $X(s)$ are contained in ROC
 - ▶ If $x(t)$ is right sided, then ROC is in the right of the rightmost pole
 - ▶ If $x(t)$ is left sided, then ROC is in the left of the leftmost pole
- ▶ If ROC includes $j\omega$ axis then $x(t)$ has FT

Inverse of Laplace Transform (LT)

- ▶ By considering σ fixed, inverse of LT can be obtained from inverse of FT:

$$\text{▶ } x(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{X(\sigma + j\omega)}_s e^{j\omega t} d\omega$$

$$\text{▶ } x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} d\omega$$

- ▶ assuming σ is fixed $\rightsquigarrow ds = j d\omega$

$$\text{▶ } \therefore x(t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} X(s) e^{st} ds$$

- ▶ If $X(s)$ is rational, we can use expanding the rational algebraic into a linear combination of lower order terms and then one may use

- ▶ $X(s) = \frac{1}{s+a} \rightsquigarrow x(t) = -e^{-at} u(-t)$ if $\text{Re}\{s\} < -a$

- ▶ $X(s) = \frac{1}{s+a} \rightsquigarrow x(t) = e^{-at} u(t)$ if $\text{Re}\{s\} > -a$

- ▶ Do not forget to consider ROC in obtaining inverse of LT!

LT Properties

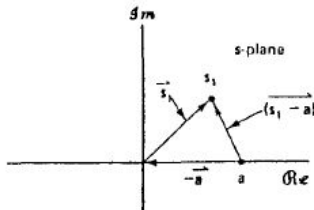
- ▶ **Linearity:** $ax_1(t) + bx_2(t) \Leftrightarrow aX_1(s) + bX_2(s)$
 - ▶ ROC contains: $R_1 \cap R_2$
 - ▶ If $R_1 \cap R_2 = \emptyset$ it means that LT does not exist
 - ▶ By zeros and poles cancelation ROC can be larger than $R_1 \cap R_2$
- ▶ **Time Shifting:** $x(t - T) \Leftrightarrow e^{-sT} X(s)$ with $\text{ROC} = R$
- ▶ **Shifting in S-Domain:** $e^{s_0 t} x(t) \Leftrightarrow X(s - s_0)$ with $\text{ROC} = R + \mathcal{R}e\{s_0\}$
- ▶ **Time Scaling:** $x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right)$ with $\text{ROC} = \frac{R}{a}$
- ▶ **Differentiation in Time-Domain:** $\frac{dx(t)}{dt} \Leftrightarrow sX(s)$ with ROC containing R
- ▶ **Differentiation in the s-Domain:** $-tx(t) \Leftrightarrow \frac{dX(s)}{ds}$ with $\text{ROC} = R$
- ▶ **Convolution:** $x_1(t) * x_2(t) \Leftrightarrow X_1(s)X_2(s)$ with ROC containing $R_1 \cap R_2$

Analyzing LTI Systems with LT

- ▶ LT of impulse response is $H(s)$ which is named **transfer function** or **system function**.
- ▶ Transfer fcn can represent many properties of the system:
 - ▶ **Causality**: $h(t) = 0$ for $t < 0 \rightsquigarrow$ It is right sided
 - ▶ **ROC of a causal system is a right-half plane**
 - ▶ Note that the converse is not always correct
 - ▶ **Example**: $H(s) = \frac{e^s}{s+1}$, $\text{Re}\{s\} > -1 \rightsquigarrow h(t) = e^{-(t+1)}u(t+1)$ it is none zero for $-1 < t < 0$
 - ▶ **For a system with rational transfer fcn, causality is equivalent to ROC being the right-half plane to the right of the rightmost pole**
 - ▶ **Stability**: $h(t)$ should be absolute integrable \rightsquigarrow its FT converges
 - ▶ **An LTI system is stable iff its ROC includes $j\omega$ axis ($0 \in \text{ROC}$)**
 - ▶ **A causal system with rational $H(s)$ is stable iff all the poles of $H(s)$ have negative real-parts (are in left-half plane)**

Geometric Evaluation of FT by Zero/Poles Plot

- ▶ Consider $X_1(s) = s - a$



- ▶ $|X_1|$: length of X_1
- ▶ $\angle X_1$: angle of X_1
- ▶ Now consider $X_2(s) = \frac{1}{s-a} = \frac{1}{X_1(s)}$
 - ▶ $\log X_2 = -\log X_1$
 - ▶ $\angle X_2 = -\angle X_1$

- For higher order fcn's:

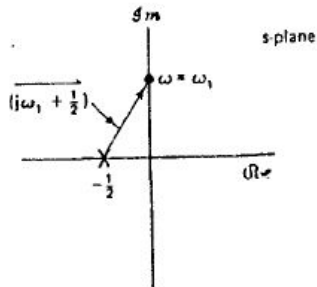
$$X(s) = M \frac{\prod_{i=1}^R (s - \beta_i)}{\prod_{j=1}^P (s - \alpha_j)}$$

- $|X(s)| = |M| \frac{\prod_{i=1}^R |s - \beta_i|}{\prod_{j=1}^P |s - \alpha_j|}$
- $\angle X(s) = \angle M + \sum_{i=1}^R \angle(s - \beta_i) - \sum_{j=1}^P \angle(s - \alpha_j)$

- Example:

$$H(s) = \frac{1/2}{s + 1/2}, \quad \text{Re}\{s\} > -\frac{1}{2}$$

- $h(t) = \frac{1}{2} e^{-t/2} u(t)$
- $s(t) = [1 - e^{-t/2}] u(t)$
- $H(j\omega) = \frac{1/2}{j\omega + 1/2}$
- $|H(j\omega)|^2 = \frac{(1/2)^2}{\omega^2 + (1/2)^2}$
- $\angle H(j\omega) = -\tan^{-1} 2\omega$
- $0 < \omega < \infty \rightsquigarrow -\pi/2 < \angle H(j\omega) < 0$
- $\omega \uparrow \rightsquigarrow |H| \downarrow, \angle H(j\omega) \downarrow$



- ▶ Now let us substitute 2 with τ in the previous example

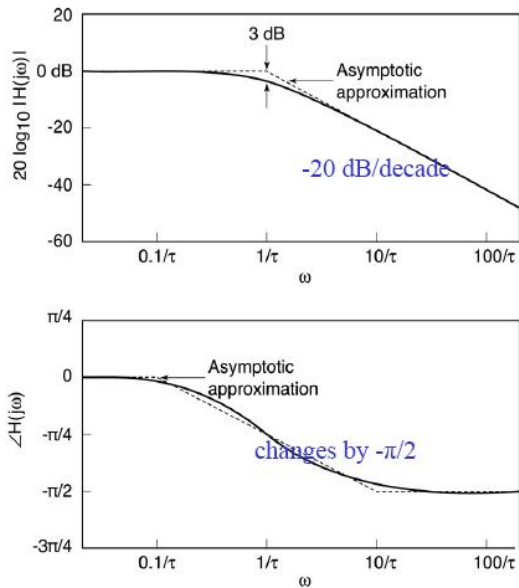
- ▶ $H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau}$

- ▶ $|H(j\omega)|^2 = \frac{(1/\tau)^2}{\omega^2 + (1/\tau)^2}, |H(j\omega)| = \begin{cases} 1 & \omega = 0 \\ \frac{1}{\sqrt{2}} & \omega = \frac{1}{\tau} \\ \frac{1}{\tau\omega} & \omega \gg \frac{1}{\tau} \end{cases}$

- ▶ $\angle H(j\omega) = -\tan^{-1} \tau\omega = \begin{cases} 0 & \omega = 0 \\ -\frac{\pi}{4} & \omega = \frac{1}{\tau} \\ -\frac{\pi}{2} & \omega \gg \frac{1}{\tau} \end{cases}$

- ▶ Relation between real part of poles and response of the systems

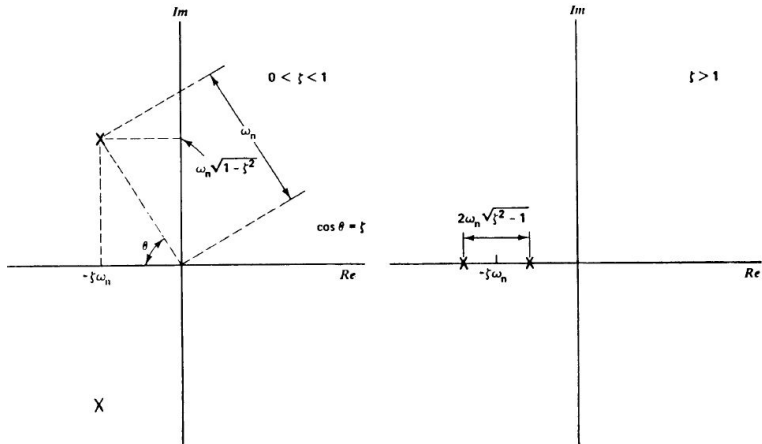
- ▶ τ is time constant of first order systems which control response speed of the systems
- ▶ Poles are located at $-\frac{1}{\tau}$
- ▶ The farther the poles from $j\omega$ axis \rightsquigarrow cut-off freq. $\uparrow, \tau \downarrow$, the faster decaying the impulse response, the faster rise time of step response



Response for Second Order system

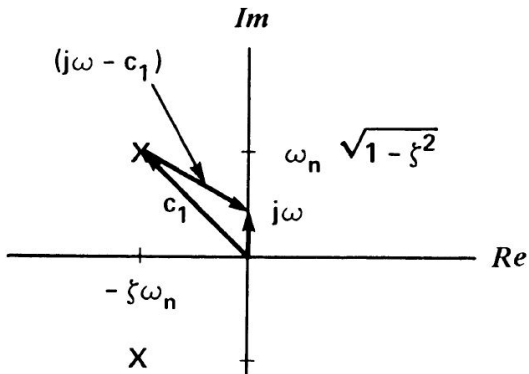
- ▶ $h(t) = M(e^{c_1 t} - e^{c_2 t})u(t)$
- ▶ $H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s - c_1)(s - c_2)}$
- ▶ $c_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
- ▶ $0 < \zeta < 1$: under damp (two complex poles), $c_2 = c_1^*$
- ▶ $\zeta = 1$ critically damp ($s = -\omega_n$)
- ▶ $\zeta > 1$: Over damp (two negative real poles)
- ▶ For fixed ω_n , $\zeta \uparrow \uparrow \rightsquigarrow$, settling time for step response \uparrow

Zero-Pole Pattern of Second Order System

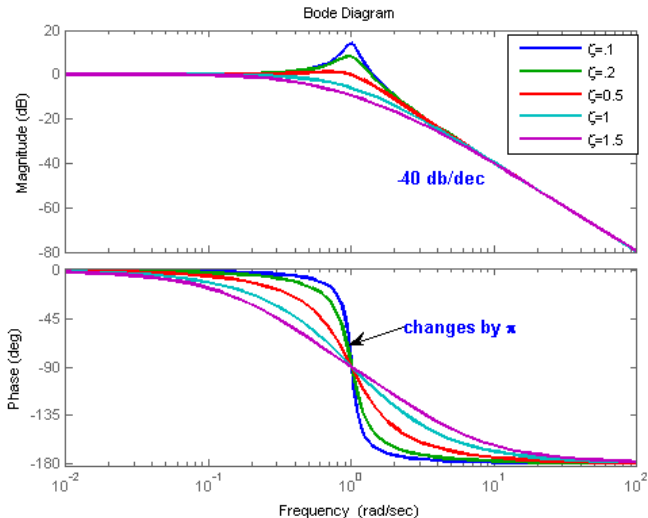


Freq. Response of Second Order System

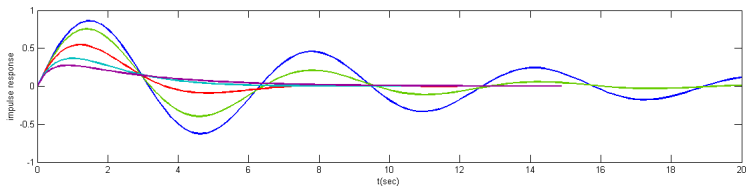
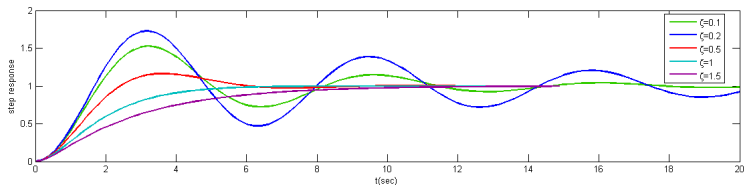
- ▶ $H(s) = \frac{\omega_n^2}{(s-c_1)(s-c_1^*)}$
- ▶ $H(j\omega) = H(s)|_{s=j\omega} = \frac{\omega_n^2}{(j\omega-c_1)(j\omega-c_1^*)}$



Bode Plot of $H(j\omega)$

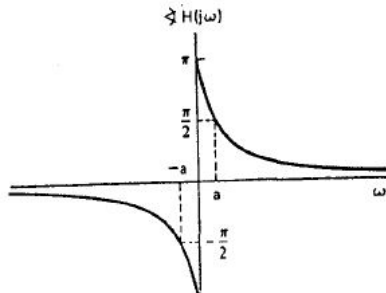
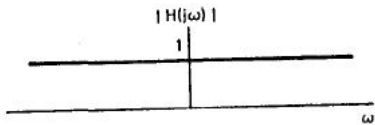
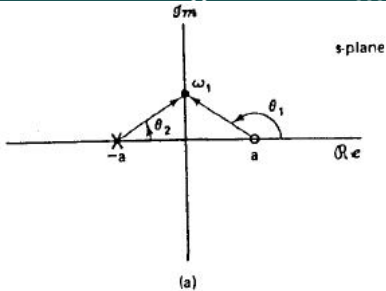


Impulse and Step Response of the second order system



All Pass Filters

- ▶ Passes the signal in all freqs. with a little decreasing/increasing the magnitude
- ▶ Why do we use all-pass filters?
- ▶ $H(s) = \frac{s-a}{s+a}$ $\text{Re}\{s\} > -a$, $a > 0$
- ▶ $|H(j\omega)| = 1$
- ▶ $\angle H(j\omega) = \theta_1 - \theta_2 = \pi - 2\theta_2 = \pi - 2\tan^{-1}\left(\frac{\omega}{a}\right) = \begin{cases} \pi & \omega = 0 \\ \frac{\pi}{2} & \omega = a \\ 0 & \omega \gg a \end{cases}$



LTI Systems Description

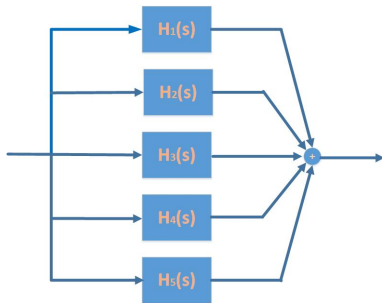
- ▶ $\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$
- ▶ $\sum_{k=0}^N a_k s^k Y(s) = \sum_{k=0}^M b_k s^k X(s)$
- ▶ $H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^M b_k s^k}{\sum_{k=0}^N a_k s^k}$
- ▶ ROC depends on
 - ▶ placement of poles
 - ▶ boundary conditions (right sided, left sided, two sided,...)

- ▶ High Order Systems can be expressed by connected simple order systems:
- ▶ Cascade Connection:



$$H(s) = H_1(s)H_2(s)H_3(s)H_4(s)H_5(s)$$

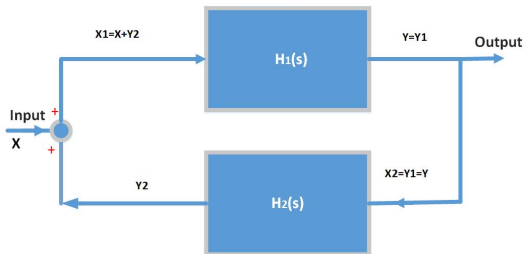
- ▶ Parallel Connection:



$$H(s) = H_1(s) + H_2(s) + H_3(s) + H_4(s) + H_5(s)$$

Feedback Interconnection of two LTI systems:

- ▶ $Y(s) = Y_1(s) = X_2(s)$
- ▶ $X_1(s) = X(s) + Y_2(s) = X(s) + H_2(s)Y(s)$
- ▶ $Y(s) = H_1(s)X_1(s) = H_1(s)[X(s) + H_2(s)Y(s)]$
- ▶ $\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 - H_2(s)H_1(s)}$
- ▶ ROC: is determined based on roots of $1 - H_2(s)H_1(s)$



Block Diagram Representation for Causal LTI Systems

- ▶ We can represent a transfer fcn by different methods:

- ▶ **Example:** $H(s) = \frac{2s^2+4s-6}{s^2+3s+2}$

1. $H(s) = (2s^2 + 4s - 6) \frac{1}{s^2+3s+2}$

2. Assuming it is causal so it is at initial rest

- ▶ $W(s) = \frac{1}{s^2+3s+2} X(s) \Leftrightarrow \frac{d^2w}{dt^2} + 3\frac{dw}{dt} + 2w = x(t)$

- ▶ $Y(s) = (2s^2 + 4s - 6)W(s) \Leftrightarrow y(t) = 2\frac{dw^2}{dt^2} + 4\frac{dw}{dt} - 6w$

3. $H(s) = 2 + \frac{6}{s+2} - \frac{8}{s+1}$

4. $H(s) = \frac{2(s-1)}{s+2} \frac{s+3}{s+1}$

Stability Analysis by Routh-Hurwitz

- ▶ b_i, c_i are defined as follows:

$$b_{n-1} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}, \quad b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_{n-1} = -\frac{1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}, \quad c_{n-3} = -\frac{1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}$$

- ▶ Follow the same rule for other rows parameters
- ▶ # of RHP root of $D(s)$ equals to # of signs changing in the first column of the table
- ▶ Necessary condition for using Routh-Horwitz method is that all coefficients of $D(s)$ should exist and have similar sign (otherwise there are more than one pole on imaginary axis, it is not stable)
- ▶ Necessary and Sufficient conditions for stability is that no signs changing appears in the first column of the Routh-Horwitz table

- ▶ **Initial Value Theorem:** If $x(t) = 0$ for $t < 0$ and $x(t)$ does not contain any impulse or higher order singularities at the origin then
$$x(0^+) = \lim_{s \rightarrow \infty} sX(s)$$
 - ▶ $X(s)$ may include a simple pole at the origin which represents a step signal.
 - ▶ More than one pole at the origin and in $j\omega$ axis make the signal oscillating
- ▶ **Final Value Theorem:** If $x(t) = 0$ for $t < 0$ and $x(t)$ is bounded when $t \rightarrow \infty$ then $x(\infty) = \lim_{s \rightarrow 0} sX(s)$
- ▶ Consider $H(s) = \frac{N(s)}{D(s)}$, n is degree of $N(s)$, d is degree of $D(s)$:
 - ▶ $H(0^+) = \begin{cases} 0 & d > n + 1 \\ \text{constant value} \neq 0 & d = n + 1 \\ \infty & d < n + 1 \end{cases}$

Unilateral LT

- ▶ It is used to describe causal systems with nonzero initial conditions:
 $\mathcal{X}(s) = \int_{0^-}^{\infty} x(t)e^{-st} dt = \mathcal{UL}\{x(t)\}$
- ▶ If $x(t) = 0$ for $t < 0$ then $\mathcal{X}(s) = X(s)$
- ▶ Unilateral LT of $x(t) =$ Bilateral LT of $x(t)u(t^-)$
- ▶ If $h(t)$ is impulse response of a causal LTI system then $H(s) = \mathcal{H}(s)$
- ▶ ROC is not necessary to be recognized for unilateral LT since it is always a right-half plane
- ▶ For rational $\mathcal{X}(s)$, ROC is in right of the rightmost pole

Similar Properties of Unilateral and Bilateral LT

- ▶ **Convolution:** Note that for unilateral LT, If **both** $x_1(t)$ and $x_2(t)$ are zero for $t < 0$, then $\mathcal{X}(s) = \mathcal{X}_1(s)\mathcal{X}_2(s)$
- ▶ **Time Scaling**
- ▶ **Shifting in s domain**
- ▶ **Initial and Finite Theorems:** they are indeed defined for causal signals
- ▶ **Integrating:** $\int_{0^-}^t x(\tau)d\tau = x(t) * u(t) \stackrel{UL}{\Leftrightarrow} \mathcal{X}(s)\mathcal{U}(s) = \frac{1}{s}\mathcal{X}(s)$
- ▶ **The main difference between \mathcal{UL} and LT is in time differentiation:**
 - ▶ $\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$
 - ▶ Use the rule $\int fdg = fg - \int gdf$
 - ▶ $\therefore \mathcal{UL}\left\{\frac{dx(t)}{dt}\right\} = s \int_{0^-}^{\infty} x(t)e^{-st} dt + x(t)e^{-st}\Big|_{0^-}^{\infty} = s\mathcal{X}(s) - x(0^-)$
 - ▶ $\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\} = s\mathcal{X}(s) - x(0^-)$
 - ▶ $\mathcal{UL}\left\{\frac{d^2x(t)}{dt^2}\right\} = \mathcal{UL}\left\{\frac{d}{dt}\left\{\frac{dx(t)}{dt}\right\}\right\} = s(s\mathcal{X}(s) - x(0^-)) - \dot{x}(0^-) = s^2\mathcal{X}(s) - sx(0^-) - \dot{x}(0^-)$
 - ▶ Follow the same rule for higher derivatives

Example

- ▶ Consider $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = x(t)$, where $y(0^-) = \beta = 3$, $\dot{y}(0^-) = \gamma = -5$, $x(t) = \alpha u(t) = 2u(t)$

- ▶ Take \mathcal{UL} :

- ▶ $s^2\mathcal{Y}(s) - \beta\mathcal{Y}(s) - \gamma + 3(s\mathcal{Y}(s) - \beta) + 2\mathcal{Y}(s) = \mathcal{X}(s)$

- ▶
$$\mathcal{Y}(s) = \underbrace{\frac{\beta(s+3) + \gamma}{s^2 + 3s + 2}}_{\text{ZIR}} + \underbrace{\frac{\mathcal{X}(s)}{s^2 + 3s + 2}}_{\text{ZSR}}$$

- ▶ **Zero State Response (ZSR):** is a response in absence of initial values

- ▶ $\mathcal{H}(s) = \frac{\mathcal{Y}(s)}{\mathcal{X}(s)}$

- ▶ **Transfer fcn is ZSR**

- ▶ ZSR: $\mathcal{Y}_1(s) = \frac{\alpha}{s(s+1)(s+2)} = \frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}$

- ▶ $y_1(t) = (1 - 2e^{-t} + e^{-2t})u(t)$

- ▶ **Zero Input Response (ZIR):** is a response in absence of input ($x(t) = 0$)

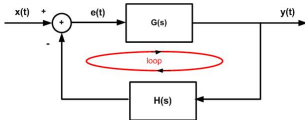
- ▶ ZIR: $\mathcal{Y}_2(s) = \frac{3(s+3) - 5}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{2}{s+2}$

- ▶ $y_2(t) = (e^{-t} + 2e^{-2t})u(t)$

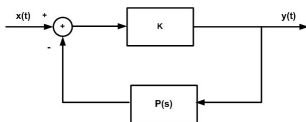
- ▶ $y(t) = y_1(t) + y_2(t)$

Feed Back Applications

- ▶ Closed loop Transfer fcn: $Q(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{\text{Open loop Gain}}{1-\text{Loop Gain}}$

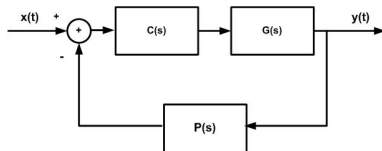


1. Inverting



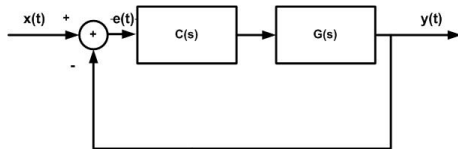
- ▶ $Q(s) = \frac{K}{1+Kp(s)}$
- ▶ If choose K s.t. $Kp(s) \gg 1$ then $Q(s) \simeq \frac{1}{p(s)}$
- ▶ Example: For a capacitor, consider i as output and v as input, it is a differentiator
- ▶ By using the above interconnection, we can make an integrator

2. Stabilizing Unstable Systems



- ▶ $G(s)$ is unstable
- ▶ We should define $P(s)$ and $C(s)$ to make closed-loop system stable (poles of closed-loop system be in LHP)
- ▶ $Q(s) = \frac{C(s)G(s)}{1+C(s)P(s)G(s)}$
- ▶ **Example 1:** $G(s) = \frac{1}{s-2}$, $C(s) = K$, $P(s) = 1$
- ▶ $Q(s) = \frac{K}{s-2+K}$
- ▶ Choosing $K > 2$ make it stable
- ▶ **Example 2:** $G(s) = \frac{1}{s^2-4}$
- ▶ By $C(s) = K$ cannot be stabilized
- ▶ Choose $C(s) = K_1 + K_2s$, $K_2 > 0$, and $K_1 > 4$ can stabilize the closed-loop system

3. Tracking



- ▶ Objective: Defining $C(s)$ s.t. $e(t) = x(t) - y(t) \rightarrow 0$ as $t \rightarrow \infty$
- ▶ $E(s) = \frac{1}{1+C(s)G(s)}X(s)$
- ▶ Consider $x(t)$ as unite step
- ▶ $\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{s}{1+C(s)G(s)} \frac{1}{s}$
- ▶ If we choose $C(s)$ s.t. $C(s)G(s) \gg 1$ then $e(t) \rightarrow 0$ as $t \rightarrow \infty$

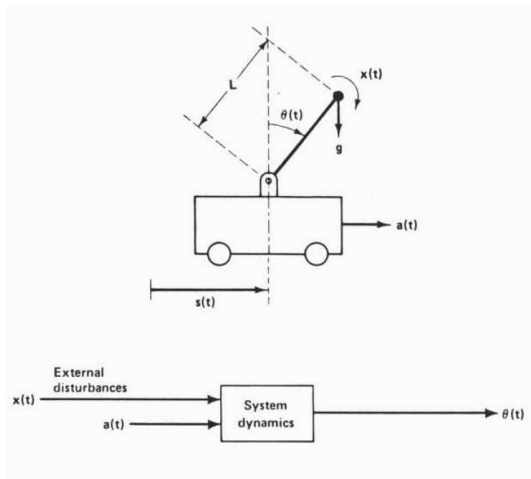
4. Decreasing effect of disturbance

5. Decreasing Sensitivity to uncertainties

Inverted Pendulum

► **Objective:**

Find proper $a(t)$ to make $\theta(t) = 0$



Inverted Pendulum

- ▶ System Dynamics:

$$L \frac{d^2\theta(t)}{dt^2} = g \sin[\theta(t)] + Lx(t) - a(t) \cos(\theta(t))$$

- ▶ Linearize it: assuming $\theta(t)$ is small

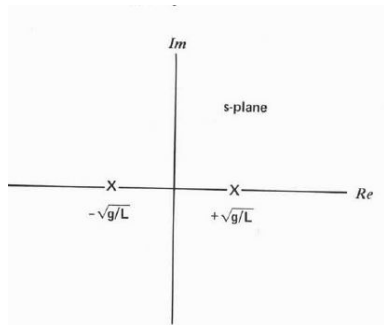
- ▶ $\sin(\theta(t)) = \theta(t)$

- ▶ $\cos(\theta(t)) = 1$

- ▶ $L \frac{d^2\theta(t)}{dt^2} = g\theta(t) + Lx(t) - a(t)$

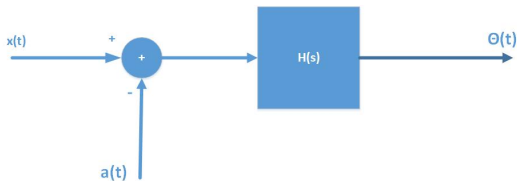
- ▶ LT:

$$\Theta(s) = \underbrace{\frac{1}{Ls^2 - g}}_{H(s)} [LX(s) - A(s)]$$



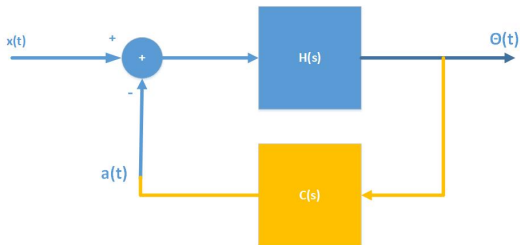
Inverted Pendulum

- ▶ $\Theta(s) = H(s)[LX(s) - A(s)]$
- ▶ $H(s) = \frac{1}{Ls^2 - g}$



Inverted Pendulum

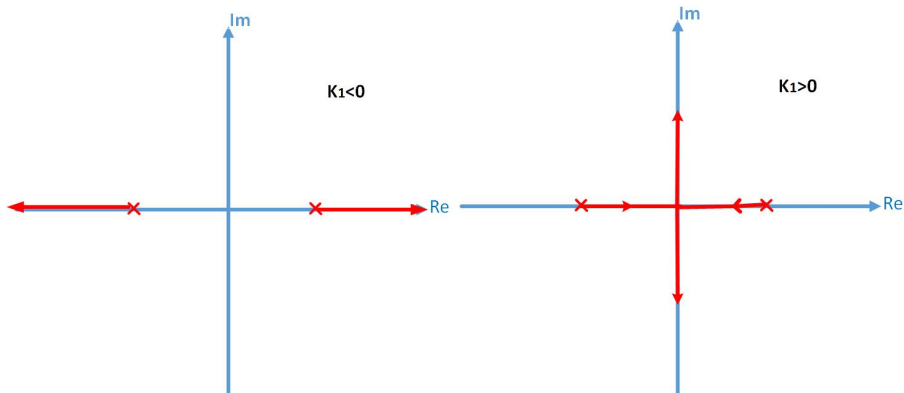
- ▶ $\Theta(s) = H(s)[LX(s) - A(s)]$
- ▶ $H(s) = \frac{1}{Ls^2 - g}$
- ▶ Using feedback connection, let us design a controller, $C(s)$ to make the pendulum in vertical position
- ▶ $\Theta(s) = \frac{LH(s)}{1 + C(s)H(s)}X(s)$



Proportional Feedback: $C(s) = K_1$

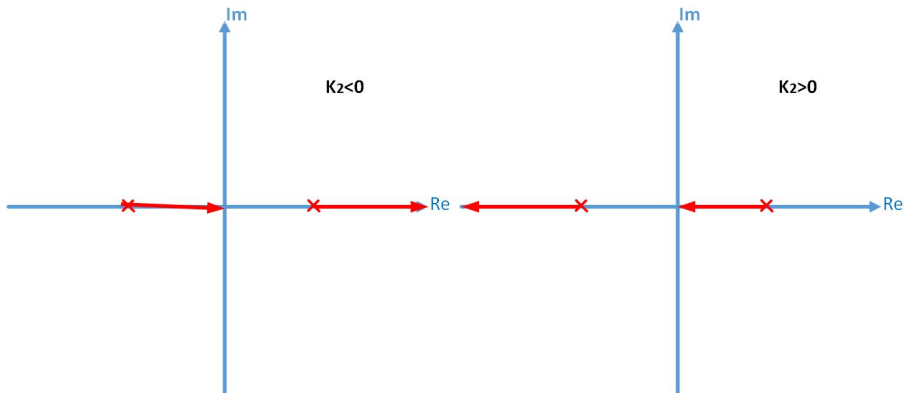
▶ $\Theta(s) = \frac{1}{s^2 - \frac{g-K_1}{L}} X(s)$

▶ Poles $s = \pm \sqrt{\frac{g-K_1}{L}}$



Derivative Feedback: $C(s) = K_2s$

- ▶ $\Theta(s) = \frac{1}{s^2 + s(K_2/L) - g/L} X(s)$
- ▶ Poles: $s = -\frac{K_2}{2L} \pm \sqrt{\left(\frac{K_2}{2L}\right)^2 + \frac{g}{L}}$



Proportional+ Derivative (PD) Feedback:

$$C(s) = K_1 + K_2s$$

- ▶ $\Theta(s) = \frac{1}{s^2 + s(K_2/L) - g/L + K_1/L} X(s)$
- ▶ Poles: $s = -\frac{K_2}{2L} \pm \sqrt{\left(\frac{K_2}{2L}\right)^2 - \frac{K_1 - g}{L}}$

