Nonlinear Control
Lecture 7: Stability Analysis IV

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Fall 2009

Input-to-State Stability
- Stability of Cascade System

Input-Output Stability
- $\mathcal{L}$ Stability of State Models

Absolute Stability
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Input-to-State Stability

- Consider the system
$$\dot{x} = f(t, x, u) \tag{1}$$
where $f : [0, \infty) \times R^m \longrightarrow R^n$ is piecewise continuous, bounded fcn $\forall t \geq 0$
- Suppose the Equ. pt. of the unforced system below is **g.u.a.s.**
$$\dot{x} = f(t, x, 0) \tag{2}$$
- What can be said about the behavior of the forced system in the presence of a bounded input $u(t)$.
- For an LTI system:
$$\dot{x} = Ax + Bu$$
where A is Hurwitz, the solution satisfies:
$$\|x(t)\| \leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k\|B\|}{\lambda} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\|$$
  - Zero-input response decays to zero
  - Zero-state response remains **bounded** for **bounded input**

Input-to-State Stability

- Can this conclusion be extended to nonlinear system (1)?
  - The answer in general is no, for instance:
$$\dot{x} = -3x + (1 + 2x^2)u$$
    when $u = 0$, the origin is **g.e.s**
  - However, with $x(0) = 2$ and $u(t) \equiv 1$, $x(t) = (3 - e^t)/(3 - 2e^t)$ is unbounded and have a finite scape time.
- View the system $\dot{x} = f(t, x, u)$ as a perturbation of the unforced system $\dot{x} = f(t, x, 0)$.
- Suppose there exists a Lyap. fcn for the unforced system and calculate $\dot{V}$ in the presence of u
- Since u is bounded, it may be possible to show that $\dot{V}$ is n.d. outside of a ball with radius μ where μ depends on $\sup \|u\|$.
- This is possible, for instance if the function $f(t, x, u)$ is Lip. in u, i.e.
$$\|f(t, x, u) - f(t, x, 0)\| \leq L\|u\|$$

Input-to-State Stability

- Having shown $\dot{V}$ is negative outside of a ball, ultimate boundedness theorem can be used, i.e.
  - $\|x(t)\|$ is bounded by a class $\mathcal{KL}$ fcn $\beta(\|x(t_0)\|, t - t_0)$ over $[t_0, t_0 + T]$ and by a class $\mathcal{K}$ fcn $\alpha^{-1}(\alpha_2(\mu))$ for $t \geq t_0 + T$
- Hence, $\|x(t)\| \ \leq \ \beta(\|x(t_0)\|, t - t_0) + \alpha^{-1}(\alpha_2(\mu)) \ \ \forall \ t \geq \ t_0$
- **Definition:** The system (1) is said to be **input-to-state stable** if there exist a class $\mathcal{KL}$ fcn β and a class $\mathcal{K}$ fcn γ s.t. for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \ \geq \ t_0$ and satisfies:
$$\|x(t)\| \ \leq \ \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)$$
  - If $u(t)$ converges to zero as $t \ \longrightarrow \ \infty$, so does $x(t)$.
  - with $u(t) \equiv 0$, the above equation reduces to:
$$\|x(t)\| \ \leq \ \beta(\|x(t_0)\|, t - t_0)$$
    implying the origin of unforced system is **g.u.a.s.**

Input-to-State Stability

- **Sufficient** condition for input-to-state stability:
- **Theorem:** *Let $V : [0,\infty) \times R^n \longrightarrow R$ be a cont. diff. fcn. s.t.*

$$\begin{aligned} \alpha_1(\|x\|) &\leq V(t,x) \leq \alpha_2(\|x\|) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) &\leq -W_3(x), \quad \forall\ \|x\| \geq \rho(\|u\|) > 0 \end{aligned}$$

$\forall (t,x,u) \in [0,\infty) \times R^n \times R^m$ where α_1 and α_2 are class $\mathcal{K}_\infty$ fcns and $W_3(x)$ is a cont. p.d. fcn. on R^n. Then, the system (1) is input-to-state stable with

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$$

Input-to-State Stability

- **Lemma:** *Suppose $f(t, x, u)$ is cont. diff. and* **globally** *Lip. in (x, u), uniformly in t. If the unforced system has a* **globally exponentially stable** *Equ. pt. at the origin, then the system (1) is* **input-to-state stable** *(ISS).*
- **Proof:**
  - View the forced system as a perturbation to unforced system
  - The converse theorem implies that the unforced system has a Lyap. fcn satisfying the **g.e.s** conditions.
  - The perturbation terms satisfies the Lip. cond. $\forall\ t \geq 0$ and $\forall\ (x, u)$.
  - Hence, $\dot{V}$ along the trajectories of forced system (1):

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, 0) + \frac{\partial V}{\partial x}[f(t, x, u) - f(t, x, 0)] \\ &\leq -c_3\|x\|^2 + c_4\|x\|L\|u\| = -c_3(1-\theta)\|x\|^2 - c_3\theta\|x\|^2 \\ &+ c_4\|x\|L\|u\|, \quad 0 < \theta < 1 \end{aligned}$$

Input-to-State Stability

$$\therefore \dot{V} \leq -c_3(1-\theta)\|x\|^2 \forall \ \|x\| \geq \frac{c_4 L\|u\|}{c_3\theta}, \quad \forall \ (t, x, u)$$

- The conditions of previous theorem are satisfied with:

$$\alpha_1(r) = c_1 r^2, \ \alpha_2(r) = c_2 r^2, \ \rho(r) = (c_4 L/c_3\theta) r$$

- $\therefore$ The system is input-to-state stable with $\gamma(r) = \sqrt{c_2/c_1}(c_4 L/c_3\theta) r$
- The previous lemma relies on **globally Lip**. fcn f and global **exponential stability** of the origin of the unforced system for **ISS**.

Input-to-State Stability

- $\dot{x} = -3x + (1 + x^2)u$ does not satisfy the global Lip. cond.
- **Example 7.1:** $$\dot{x} = -\frac{x}{1 + x^2} + u = f(x, u)$$

  has a globally Lip. f since $\frac{\partial f}{\partial x} = -\frac{1-x^2}{(1+x^2)^2}$ and $\frac{\partial f}{\partial u} = 1$ and are globally bounded.
- The origin of unforced system $\dot{x} = -\frac{x}{1+x^2}$ is $g.a.s(V = x^2/2 \implies \dot{v} = -\frac{x^2}{1+x^2}$ **n.d.** for all x)
- The system is locally **e.s** because of the linearized system $\dot{x} = -x$
- However, the system is not **g.e.s**

$$u \equiv 1, f(x, u) \geq 1/2 \implies x(t) \geq x(t_0) + t/2 \ \forall \ t \geq t_0$$

- If **g.e.s.** and globally Lip. conds. are not satisfied, then we can use previous theorem to show **ISS** (i.e. find a region $\|x\| \geq \mu$ in which $\dot{V} < 0$)

Input-to-State Stability

- **Example 7.2:** $$\dot{x} = -x^3 + u$$

  has a **g.a.s.** Equ. pt. at the origin when $u = 0$.
- Let $v = x^2/2$, then $\dot{V}$ can be written as:
  $\dot{V} = -x^4 + ux \;=\; -(1-\theta)x^4 - \theta x^4 + xu$
  $\leq \; -(1-\theta)x^4, \; \forall |x| \; \geq \; \left(\frac{|u|}{\theta}\right)^{1/3} \quad 0 \; < \; \theta \; < \; 1.$
- The system is ISS with $\gamma(r) = (r/\theta)^{1/3}$
- **Example 7.3:** $\dot{x} = f(x,u) = -x - 2x^3 + (1+x^2)u^2$

  has a **g.e.s.** Equ. pt. at the origin when $u = 0$.
- However, f is not globally Lip. Let $v = x^2/2$, then:
  $$\dot{V} = -x^2 - 2x^4 + x(1+x^2)u^2 \;\leq\; -x^4, \; \forall \; |x| \; \geq \; u^2$$
- The system is ISS with $\gamma(r) = r^2$

Stability of Cascade System

- Consider the cascade system

$$\dot{x}_1 = f_1(t, x_1, x_2), \quad f_1 : [0, \infty) \times R^{n_1} \times R^{n_2} \longrightarrow R^{n_1} \qquad (3)$$
$$\dot{x}_2 = f_2(t, x_2), \quad f_2 : [0, \infty) \times R^{n_2} \longrightarrow R^{n_2} \qquad (4)$$

where f_1 and f_2 are p.c. in t and locally Lip. in $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

- Suppose $\dot{x}_1 = f_1(t, x_1, 0)$ and (4) both have **g.u.a.s.** Equ. pt. at $x_1 = 0$ and $x_2 = 0$.
- Under what condition the origin of the cascade system is also **g.u.a.s.**?
- The condition is that (3) should be **ISS** with x_2 viewed as input.
- **Lemma:** *Under the assumption given above, if the system (3) with x_2 as input, is* **ISS** *and the origin of (4) is* **g.u.a.s.**, *then the origin of the cascade system (3) and (4) is* **g.u.a.s.**

Input-Output Stability

- The foundation of input-output (I/O) approaches to nonlinear systems can be found in 1960's by Sandberg and Zames
- An input-output model relates output to input with no knowledge of the internal structure (state equation).

$$y = Hu, \quad u : [0, \infty) \rightarrow \mathcal{R}^m$$

- The norm function $\|u\|$ should satisfy the three properties
  1. $\|u\| = 0$ iff $u = 0$ and it is strictly positive otherwise
  2. scaling property $\forall a > 0, u \Rightarrow \|au\| = a\|u\|$
  3. triangular inequality: $\forall u_1, u_2,\ \|u_1 + u_2\| \leq \|u_1\| + \|u_2\|$
- Example:

$$\begin{aligned}
\|u\|_{\mathcal{L}_\infty^m} &= \sup_{t\geq 0} \|u\| < \infty \\
\|u\|_{\mathcal{L}_2^m} &= \sqrt{\int_0^\infty u^T(t)u(t)dt} < \infty \\
\|u\|_{\mathcal{L}_p^m} &= \int_0^\infty \|u\|^p dt)^{1/p} < \infty, \quad 1 \leq p < \infty
\end{aligned}$$

Input-Output Stability

- Stable system: any "well-behaved" input generate a "well-behaved" output
- Extended space: $\mathcal{L}_e^m = \{u | u_\tau \in \mathcal{L}^m, \forall \tau \in [0, \infty)\}$
  - where u_τ is a truncation of u: $u_\tau(t) = \begin{cases} u(t) & 0 \leq t \leq \tau \\ 0 & t > \tau \end{cases}$
  - It allows us to deal with unbounded "ever-growing" signals
  - **Example:** $u(t) = t \notin \mathcal{L}_\infty$ but $u_\tau(t) \in \mathcal{L}_{\infty e}$
- Casuality: mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is causal if the output $(Hu)(t)$ at any time t depends only on the value of the input **up to time** t

$$(Hu)_\tau = (Hu_\tau)_\tau$$

Input-Output Stability

- Definition: *A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is $\mathcal{L}$* **stable** *if there exist a class $\mathcal{K}$ function α, defined on $[0,\infty)$ and a nonneg const. β s.t.*

$$\|(Hu)_\tau\|_\mathcal{L} \le \alpha(\|u_\tau\|_\mathcal{L}) + \beta, \quad \forall u \in \mathcal{L}_e^m, \tau \in [0,\infty) \tag{5}$$

It is **finite-gain** *$\mathcal{L}$ stable if there exist nonneg. const. γ and β s.t.*

$$\|(Hu)_\tau\|_\mathcal{L} \le \gamma(\|u_\tau\|_\mathcal{L}) + \beta, \quad \forall u \in \mathcal{L}_e^m, \tau \in [0,\infty) \tag{6}$$

  - β is bias term $\rightsquigarrow$ allows Hu does not vanish at $u = 0$
  - In finite-gain $\mathcal{L}$ stability , the smallest possible γ is desired to satisfy (6)
  - $\mathcal{L}_\infty$ stability is bounded-input-bounded-output stability.
- **Example 7.4:**

  $y(t) = h(u) = a + b\tanh cu = a + b\frac{e^{cu}-e^{-cu}}{e^{cu}+e^{-cu}}$, for $a, b, c \ge 0$
  - using the fact: $\acute{h}(u) = \frac{4bc}{(e^{cu}+e^{-cu})^2} \le bc, \quad \forall\ u \in R$
  - $\therefore\ |h(u)| \le a + bc|u|, \ \forall\ u \in R$
  - it is finite gain $\mathcal{L}_\infty$ stable with $\gamma = bc, \quad \beta = a$

Input-Output Stability

- **Example 7.5:** $y(t) = h(u) = u^2$
  - $\sup_{t\geq 0} |h(u(t))| \leq (\sup_{t\geq 0} |u(t)|)^2$
  - $\therefore$ it is $\mathcal{L}_\infty$ stable with $\beta = 0, \ \alpha(r) = r^2$
  - But it is **not** finite-gain $\mathcal{L}_\infty$ stable since $h(u)$ can not be bounded by a straight line of the form $|h(u)| \leq \gamma|u| + \beta$ for all $u \in R$
- **Example 7.6:** $y = \tan u$
  - $y(t)$ is defined only for $|u(t)| < \frac{\pi}{2}, \ \forall t \geq 0 \rightsquigarrow$ it is not $\mathcal{L}_\infty$ stable
  - If we restrict $|u| \leq r \leq \frac{\pi}{2} \rightsquigarrow |y| \leq \left(\frac{\tan r}{r}\right)|u|$
  - it is small-gain $\mathcal{L}$ stable
- Definition: *mapping $H : \mathcal{L}_e^m \rightarrow \mathcal{L}_e^q$ is* **small-signal $\mathcal{L}$ stable**/**small-signal finite-gain $\mathcal{L}$ stable** *if there exist r s.t. inequality (5)/(6) is satisfied for all $u \in \mathcal{L}_e^m$ with $\sup_{0\leq t\leq \tau} \|u\| \leq r$*

$\mathcal{L}$ Stability of State Models

- What can we say about I/O stability based o the formalism of Lyapunov stability?
- Consider

$$\begin{aligned} \dot{x} &= f(t, x, u) \\ y &= h(t, x, u) \end{aligned} \tag{7}$$

  - $x \in R^m, \ y \in R^q$
  - $f : [0, \infty) \times D \times D_u \to R^n$ is p.c. in t, locally Lipshitz in (x, u)
  - $h : [0, \infty) \times D \times D_u \to R^q$ p.c. in t and cont. in (x, u)
  - $D \subset R^n$ is a domain containing $x = 0$
  - $D_u \subset R^m$ is a domain containing $u = 0$
  - Assume the unforced system $\dot{x} = f(t, x, 0)$ is u.a.s (or u.e.s)

- Theorem: *Consider the system (7) and take $r_u, r > 0$ s.t. $\{\|x\| \leq r\} \subset D$ and $\{\|u\| \leq r_u\} \subset D_u$. Suppose that*
  - *$x = 0$ is an e.s. Equ. point of $\dot{x} = f(t, x, 0)$ and there is a Lyap. fcn $V(t, x)$ and positive const c_i, $i = 1, \ldots, 4$ that*

$$c_1 \|x\|^2 \leq V(t,x) \leq c_2 \|x\|^2, \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3 \|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4 \|x\| \qquad \forall (t,x) \in [0,\infty) \times D,$$

  - *$\forall (t,x,u) \in [0,\infty) \times D \times D_u$ and for some nonneg. const. L, η_1, and η_2:*

$$\|f(t,x,u) - f(t,x,0)\| \leq L\|u\|, \quad \|h(t,x,u)\| \leq \eta_1 \|x\| + \eta_2 \|u\|$$

- *Then for each $\|x_0\| \leq r\sqrt{c_1/c_2}$ the system is small-signal finite-gain $\mathcal{L}_p$ stable for each $p \in [1,\infty]$. In particular, for each $u \in \mathcal{L}_{pe}$ with $\sup_{0 \leq t \leq \tau} \|u\| \leq \min\{r_u, c_1 c_3 r/(c_2 c_4 L)\}$ the output satisfies:*

$$\|y_\tau\|_{\mathcal{L}_q} \leq \gamma \|u_\tau\|_{\mathcal{L}_p} + \beta, \quad \tau \in [0,\infty)$$

$$\gamma = \eta_2 + \frac{\eta_1 c_2 c_4 L}{c_1 c_3}, \ \beta = \eta_1 \|x_0\| \sqrt{\frac{c_2}{c_1}} \rho, \ \rho = \begin{cases} 1, & p = \infty \\ \left(\frac{2c_2}{c_3 p}\right)^{1/p}, & p \in [1,\infty) \end{cases}$$

$\mathcal{L}$ Stability of State Models

- Theorem Cont'd. *If the origin is g.e.s and all assumptions hold for globally (with $D = R^n$ and $D_u = R^m$), then for each $x_0 \in R^n$ the system is finite-gain $\mathcal{L}_p$ stable for each $p \in [1, \infty]$*
- Exercise: Provide similar conditions for finite-gain $\mathcal{L}_p$ stability of LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- **Example 7.7:**

$$\begin{aligned} \dot{x} &= -x - x^3 + u, \quad x(0) = x_0 \\ y &= tanhx + u \end{aligned}$$

  - The origin of $\dot{x} = -x - x^3$ is g.e.s. (Use Lyap $V(x) = x^2/2$)
  - $c_1 = c_2 = 1/2, \quad c_3 = c_4 = 1$
  - $L = \eta_1 = \eta_2 = 1$
  - The system is finite-gain $\mathcal{L}_p$ stable

$\mathcal{L}$ Stability of State Models

- **Example 7.8:**
$$\begin{aligned}\dot{x}_1 &= -x_2\\ \dot{x}_2 &= -x_1 - x_2 - a\tanh x_1 + u, \quad a \geq 0\\ y &= x_1\end{aligned}$$

  - For unforced system, take $V(x) = x^T P x = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$
  - $\dot{V} = -2p_{12}(x_1^2 + ax_1\tanh x_1) + 2(p_{11} - p_{12} - p_{22})x_1x_2 - 2ap_{22}x_2\tanh x_1 - 2(p_{22} - p_{12})x_2^2$
  - To cancel the cross product term x_1x_2, choose $p_{11} = p_{12} + p_{22}$
  - To make P p.d., choose $p_{22} = 2p_{12} = 1$
  - Use the facts: $x_1 \tanh x_1 > 0, \forall x_1 \in R$, $|x_1| \geq |\tanh x_1|$, and $x_1^2 + x_2^2 \geq 2a|x_1||x_2|$
  - $\rightsquigarrow \dot{V} = -x_1^2 - x_2^2 - ax_1\tanh x_1 - 2ax_2\tanh x_1 \leq -\|x\|_2^2 + 2ax|x_1||x_2|$
  - $\therefore$ for all $a < 1$, $\dot{V} < 0$
  - $c_1 = \lambda_{min}(P)$, $c_2 = \lambda_{max}(P)$, $c_3 = 1 - a$ and $c_4 = 2\|P\|_2 = 2\lambda_{max}(P)$
  - $L = \eta_1 = 1$, $\eta_2 = 0$
  - All conditions are satisfied globally $\rightsquigarrow$ system is finite-gain $\mathcal{L}_P$ stable

$\mathcal{L}$ Stability of State Models

- Theorem: *Consider the system (7) and take $r_u, r > 0$ s.t. $\{\|x\| \leq r\} \subset D$ and $\{\|u\| \leq r_u\} \subset D_u$. Suppose that*
  - *$x = 0$ is an a.s. Equ. point of $\dot{x} = f(t, x, 0)$ and there is a Lyap. fcn $V(t, x)$ and class $\mathcal{K}$ fcns $\alpha_i, \ i = 1, ..., 4$ that*

$$\alpha_1(\|x\|) \ \leq \ V(t,x) \ \leq \ \alpha_2(\|x\|), \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \ \leq \ -\alpha_3(\|x\|)$$

$$\frac{\partial V}{\partial x} \ \leq \ \alpha_4(\|x\|) \qquad \forall (t,x) \ \in \ [0,\infty) \ \times \ D,$$

  - *$\forall (t, x, u) \in [0, \infty) \times D \times D_u$ and for some class $\mathcal{K}$ $\alpha_i, \ i = 5, .., 7$, nonneg conts. η:*

$$\|f(t,x,u) - f(t,x,0)\| \leq \alpha_5(\|u\|), \quad \|h(t,x,u)| \leq \alpha_6(\|x\|) + \alpha_7(\|u\|) + \eta$$

- *Then for each $\|x_0\| \leq \alpha_2^{-1}(\alpha_1(r))$ the system is small-signal finite-gain $\mathcal{L}_\infty$ stable*

$\mathcal{L}$ Stability of State Models

- **Theorem**: *Consider the system (7) with $D = R^n$ and $D_u = R^m$. Suppose that*
  - *The system is ISS.*
  - *for all $(t, x, u) \in [0, \infty) \times R^n \times R^m$, some class $\mathcal{K}$ fcns α_1, α_2 and a const. $\eta \geq 0$*

$$\|h(t, x, u)\| \leq \alpha_6(\|x\|) + \alpha_7(\|u\|) + \eta$$

- *Then for each $x_0 \in R^n$, the system is $\mathcal{L}_\infty$ stable.*
- **Example 7.9:**

$$\begin{aligned} \dot{x} &= -x - 2x^3 + (1 + x^2)u^2 \\ y &= x + u \end{aligned}$$

  - $V = x^2/2 \rightsquigarrow \dot{V} = -x^2 - 2x^4 + x(1 + x^2)u^2 \leq -x^4 \quad \forall |x| \geq u^2$
  - $\therefore$ the system is ISS with $\gamma = r^2$
  - $\alpha_6 = r^2, \;\; \alpha_7 = r$ and $\eta = 0$
  - Therefore the system is $\mathcal{L}_\infty$ stable

$\mathcal{L}$ Stability of State Models

- **Example 7.10:**

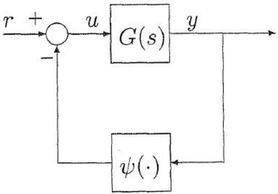
$$\begin{aligned} \dot{x}_1 &= -x_1^3 + g(t)x_2 \\ \dot{x}_2 &= -g(t)x_1 - x_2^3 + u \\ y &= x_1 + x_2 \end{aligned}$$

g(t) is continuous and bounded for $t \geq 0$

- $V = x_1^2 + x_2^2$
- $\dot{V} = -2x_1^4 - 2x_2^2 + 2x_2u$
- $2(x_1^4 + x_2^4) \geq \|x\|_2^4$ and $0 < \theta < 1 \rightsquigarrow \dot{V} \leq -(1-\theta)\|x\|_2^4, \forall \|x\|_2 \geq (\frac{2|u|}{\theta})^{1/3}$
- globally: $\alpha_1(r) = \alpha_2(r) = r^2$, $W_3(x) = -(1-\theta)\|x\|_2^4$ and $\rho(r) = (2r/\theta)^{1/3}$
- $\therefore$ The system is globally ISS
- globally: $\alpha_6(r) = \sqrt{2}r$, $\alpha_7(r) = 0$, and $\eta = 0$
- Therefore the system is $\mathcal{L}_\infty$ stable

Absolute Stability

- Many nonlinear systems are composed of a nonlinear element in feedback connection with a linear system
- Examples include actuator/sensor saturation, relay, and other hard nonlinearities.
- If the transfer fcn of the linear subsystem is **positive real**, (PR), it may lead to generation of a Lyap. fcn for the whole system.
- After introducing positive real (PR) and strictly positive real (SPR) functions, some frequency-domain sufficient conditions will be discussed for absolute stability in the form of SPR of some transfer functions.



Feedback connection.

Positive Real Systems

- Consider the n^{th}-order SISO system:

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_0}, \quad n \geq m$$

- Definition: *A transfer function $G(s)$ is* ***positive real****, if*

$$Re[G(s)] \geq 0 \;\; \text{for all} \;\; Re[s] \geq 0$$

it is ***strictly positive real*** *if $G(s-\epsilon)$ is positive for some $\epsilon > 0$*

- The condition means that $G(s)$ maps every point in RHP of s-plane (including the $j\omega$ axis) to RHP of the $G(s)$ plane.

Positive Real Systems

- **Example 7.11:**

$$
\begin{aligned}
G(s) &= \frac{1}{s+\lambda}, \quad \lambda > 0 \\
G(\sigma + j\omega) &= \frac{1}{\sigma + j\omega + \lambda} = \frac{\sigma + \lambda - j\omega}{(\sigma+\lambda)^2 + \omega^2}
\end{aligned}
$$

  - $\therefore Re[G(s)] \geq 0$ if $\sigma \geq 0 \implies G(s)$ is positive real.
  - It is also SPR, e.g. by selecting $\epsilon = \lambda/2$
- It is not always easy to use the definition for higher order systems.
- **Theorem:** *A transfer function $G(s)$ is strictly positive real iff*
  1. *$G(s)$ is a strictly stable transfer function*
  2. *The real part of $G(s)$ is strictly positive along the $j\omega$ axis, i.e*
     $\forall \omega \geq 0 \qquad Re[G(j\omega)] > 0$

Positive Real Systems

- The theorem implies some necessary conditions:
  - $G(s)$ is a strictly stable
  - The Nyq. plot lies entirely in RHP, the phase is always less than 90°
  - $G(s)$ has relative degree 0 or 1.
  - $G(s)$ is strictly minimum phase (all zeros are strictly in LHP)

$$\begin{aligned}
G_1(s) &= \frac{s-1}{s^2+as+b} \\
G_2(s) &= \frac{s+1}{s^2-s+1} \\
G_3(s) &= \frac{1}{s^2+as+b} \\
G_4(s) &= \frac{s+1}{s^2+s+1}
\end{aligned}$$

- G_1 is not SPR since it is nonminimum phase, G_2 is not SPR since it is unstable, G_3 is not SPR since its relative degree is 2,

Positive Real Systems

$$
\begin{aligned}
G_4(j\omega) &= \frac{j\omega + 1}{-\omega^2 + j\omega + 1} \\
Re[G_4(j\omega)] &= \frac{1}{(1-\omega^2)^2 + \omega^2} \quad > 0 \;\; \forall\, \omega
\end{aligned}
$$

which shows that G_4 is SPR since it is also strictly stable.

- The basic difference between PR and SPR functions is that PR fcns can have poles on the $j\omega$ axis while SPR fcns cannot. For instance

$$G(s) = \frac{1}{s} \;=\; \frac{\sigma - j\omega}{\sigma^2 + \omega^2}$$

  is clearly PR but not SPR.

- **Theorem:** *A transfer function $G(s)$ is positive real iff*
  1. *$G(s)$ is a stable transfer function*
  2. *The poles of $G(s)$ on the $j\omega$ are simple and the associated residues are real and non-negative.*

Positive Real Transfer Matrices

- The extension of positive real transfer functions for MIMO systems
- **Definition:** A $p \times p$ proper rational transfer function matrix $G(s)$ is called positive real if
  1. Poles of all elements of $G(s)$ are in $Re[s] \leq 0$
  2. $G(j\omega) + G^T(-j\omega)$ is positive semi-definite for any $\omega \geq 0$ s.t. $j\omega$ is not a pole of $G(s)$.
  3. The poles of $G(s)$ on the $j\omega$ are simple and the associated residue matrices are positive semi-definite Hermitian.

  It is SPR if $G(s-\epsilon)$ is PR for some $\epsilon > 0$

Kalman-Yakubovich Lemma

- Many variations of Kalman-Yakubovich lemma exist:
  - **Kalman-Yakubovich (Positive Real) Lemma**: *Let $G(s) = C(sI - A)^{-1}B + D$ be a $p \times p$ TF where (A, B) is controllable and (C, A) is observable. Then $G(s)$ is* **PR** *if and only if there exist matrices $P = P^T > 0$, L, and W s.t.*

$$\begin{aligned} PA + A^T P &= -L^T L \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

  - **Kalman-Yakubovich-Popov Lemma**: *Let $G(s) = C(sI - A)^{-1}B + D$ be a $p \times p$ TF where (A, B) is controllable and (C, A) is observable. Then $G(s)$ is* **SPR** *if and only if there exist matrices $P = P^T > 0$, L, and W, and a positive constant ϵ s.t.*

$$\begin{aligned} PA + A^T P &= -L^T L - \epsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

- Consider the feedback system shown in Fig. and assume $r = 0$.
- The unforced system is represented by

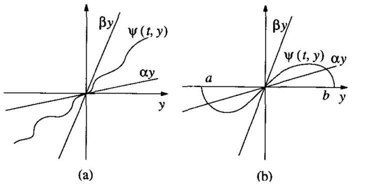
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ u &= -\psi(t, y) \end{aligned} \tag{8}$$

  where
  - $x \in R^n$, $u, y \in R^p$, (A, B) is controllable,
  - (A, C) is observable,
  - $\psi : [0, \infty) \times R^p \longrightarrow R^p$ is a memoryless, possibly time-varying nonlinearity, p.c. in t and locally Lip. in y.
- Assume the feedback system $u = \psi(t, Cx + Du)$ has a unique solution u for every (t, x) which is always the case when $D = 0$.
- Controllability and observability assumptions ensure that $\{A, B, C, D\}$ is a minimal realization.

- The nonlinearity $\psi(.,.)$ is required to satisfy a **sector condition**:
- For a SISO system, $\psi: [0,\infty) \times R \longrightarrow R$ satisfies a sector condition if $\exists\ \alpha,\ \beta,\ a\ \&\ b\ (\beta > \alpha,\ a < 0 < b)$ s.t.

$$[\psi(t,y) - \alpha y][\psi(t,y) - \beta y] \leq 0,\ \forall y\ \in [a,b]$$

- If it holds $\forall y\ \in [-\infty,\infty]$, the sector condition holds globally.
- We also say ψ belongs to a sector $[\alpha,\beta],\ (\alpha,\beta],[\alpha,\beta),\ (\alpha,\beta)$
- For MIMO: $\psi(t,y) = \begin{bmatrix} \psi_1(t,y_1) \\ \vdots \\ \psi_p(t,y_p) \end{bmatrix}$ where $\psi_i(t,y)$ satisfies the above sector condition with $\alpha_i,\ \beta_i,\ a_i,\ b_i$.



(a) Global sector nonlinearities; (b) local sector nonlinearities.

Absolute Stability

- Taking $K_{min} = diag(\alpha_1, \ \ldots, \ \alpha_p)$, $K_{max} = diag(\beta_1, \ \ldots, \ \beta_p)$, and $\Gamma = \{y \ \in \ R^p \mid a_i \ \leq \ y_i \ \leq \ b_i\}$, then

$$[\psi(t,y) - K_{min}y]^T[\psi(t,y) - K_{max}y] \ \leq \ 0 \quad \forall t \ \geq \ 0, \ \forall \ y \in \ \Gamma$$

- **Definition:** *A memoryless nonlinearity $\psi : \ [0,\infty) \ \times \ R^p \ \longrightarrow \ R^P$ is said to satisfy the sector condition if:*

$$[\psi(t,y) - K_{min}y]^T[\psi(t,y) - K_{max}y] \ \leq \ 0 \quad \forall t \ \geq \ 0, \ \forall \ y \in \ \Gamma$$

  for some real matrices K_{min} and K_{max} where $K = K_{max} - K_{min}$ is a symmetric **p.d.** *matrix and the interior of Γ is connected and contains the origin. If $\Gamma = R^p$, then ψ satisfies the sector condition globally in which case ψ is said to belong to a sector $[K_{min}, \ K_{max}]$ or $(K_{min}, \ K_{max})$.*

- The objective is to show that $x = 0$ is **a.s.** Eq. pt. for **all** nonlinearities in the sector. Such system is said to be **absolutely stable**

Absolute Stability

- **Definition:** *Consider the system (9) where ψ satisfies the sector condition. The system is absolutely stable if $x = 0$ is* **g.u.a.s** *for any nonlinearity in the given sector. It is* **absolutely stable with a finite domain** *if the origin is* **u.a.s.**.
- **Circle Criterion:**
  - Consider system (9) and let A be Hurwitz & ψ satisfies the sector condition with $K_{min} = 0$, i.e

$$\psi(t, y)^T(\psi(t, y) - Ky) \ \leq \ 0 \ \ \forall t \ \geq \ 0, \ \forall \ y \in \ \Gamma \subset R^p$$

  - Let $V(x) = x^T Px, \ P = P^T > 0$ to be chosen.

$$\dot{V} = x^T(PA + A^T P)x - 2x^T PB\psi(t, y)$$

Circle Criterion

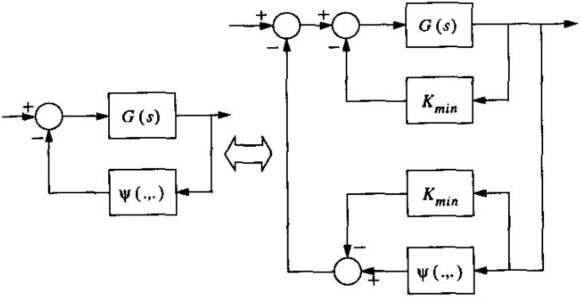
- Since $-2\psi^T(\psi - Ky) \geq 0 \quad \therefore$
$$\begin{aligned} \dot{V} &\leq x^T(PA + A^TP)x - 2x^TPB\psi(t,y) - 2\psi^T(t,y)(\psi(t,y) - Ky) \\ &= x^T(PA + A^TP)x + 2x^T(C^TK - PB)\psi - 2\psi^T\psi \end{aligned}$$

- Suppose, there are matrices $P = P^T > 0$ & L and a constant $\epsilon > 0$ s.t.
$$PA + A^TP = -L^TL - \epsilon P, \quad PB = C^TK - \sqrt{2}L^T$$
$$\begin{aligned} \therefore \ \dot{V} &\leq -\epsilon x^TPx - 2x^TL^TLx + 2\sqrt{2}x^TL^T\psi - 2\psi^T\psi \\ &= -\epsilon x^TPx - [Lx - \sqrt{2}\psi]^T[Lx - \sqrt{2}\psi]^T \\ &\leq -\epsilon x^TPx < 0 \end{aligned}$$

- $\dot{V}$ is **n.d.** if we can find P, L, , and ϵ satisfying above equations
- This is the case iff $Z(s) = I + KC(sI - A)^{-1}B$ is SPR, according to KYP lemma. Since (A, C) is obs. $\implies$ (A, KC) is obs. since K is nonsingular

Circle Criterion

- **Lemma:** *Consider system (9), where A is Hurwitz, (A, B) is controllable, (A, C) is observable & ψ satisfies the sector condition globally, then the system is* **absolutely stable** *if $Z(s) = I + KC(sI - A)^{-1}B$ is* **SPR**.
- The condition on having A Hurwitz can be removed by applying the loop transformation as shown in Fig.



Circle Criterion

- The new linear TF is:

$$\begin{aligned} G_T(s) &= G(s)[I + K_{min}G(s)]^{-1} \quad \text{or, equivalently} \\ \dot{x} &= (A - BK_{min}C)x + Bu \\ y &= Cx \quad \text{with} \\ \psi_T(t,y) &= \psi(t,y) - K_{min}y \end{aligned}$$

- Now, if ψ satisfies the sector condition, then ψ_T satisfies the sector condition with $K = K_{max} - K_{min}$.

$$\begin{aligned} (\psi - K_{min}y)^T(\psi - K_{max}y) &\leq 0, \quad \psi = \psi_T + K_{min}y \implies \\ \psi_T^T(\psi_T + (K_{min} - K_{max})) &= \psi_T^T(\psi_T - K) \leq 0 \end{aligned}$$

- If $(A - BK_{min}C)$ is Hurwitz $\implies$ system is absolutely stable if $Z_T(s) = I + KG_T(s)$ is SPR.

Circle Criterion

- Note that

$$
\begin{aligned}
Z_T(s) &= I + (K_{max} - K_{min})G(s)(I + K_{min}G(s))^{-1} \\
&= (I + K_{max}G(s))(I + K_{min}G(s))^{-1}
\end{aligned}
$$

- **Theorem:** *Consider the system (9) where (A, B) is cont. and (A, C) is obs. and ψ satisfies the sector condition globally. Then, the system is* **absolutely stable** *if*

$$G_T(s) = G(s)[I + K_{min}G(s)]^{-1}$$

is Hurwitz and $Z_T(s) = (I + K_{max}G(s))(I + K_{min}G(s))^{-1}$ is SPR.

- The theorem is known as **multivariable circle criterion.**

Example 7.12

- Consider system (9). Let $G(s)$ is Hurwitz and

$$\gamma_1 = \sup_{\omega \in R} \sigma_{max}[G(j\omega)] = \sup_{\omega \in R} \|[G(j\omega)]\|$$

where σ_{max} denotes the max. singular value, γ_1 is finite since $G(s)$ is Hurwitz.

- Suppose the nonlinearity satisfies

$$\|\psi(t, y)\| \ \leq \ \gamma_2 \|y\|, \forall t \geq 0, \ \forall y \in R^p.$$

then it satisfies the sector condition with $K_{min} = -\gamma_2 I$ and $K_{max} = \gamma_2 I$.

- We now need to show that

$$\begin{aligned} G_T(s) &= G(s)[I - \gamma_2 G(s)]^{-1} \text{ is Hurwitz and} \\ Z_T(s) &= (I + \gamma_2 G(s))(I - \gamma_2 G(s))^{-1} \text{ is SPR} \end{aligned}$$

Example Cont'd

- We have
$$\begin{aligned} det(G) \neq 0 \quad &\Longleftrightarrow \quad \sigma_{min}(G) > 0 \\ \sigma_{max}(G^{-1}) \quad &= \quad \frac{1}{\sigma_{min}(G)} \qquad \text{if } \sigma_{min} > 0 \\ \sigma_{min}(I + G) \quad &\geq \quad 1 - \sigma_{max}(G) \\ \sigma_{max}(G_1 G_2) \quad &\leq \quad \sigma_{max}(G_1)\, \sigma_{max}(G_2) \end{aligned}$$
- $\therefore\ \sigma_{min}[I - \gamma_2 G(j\omega)]) \geq 1 - \gamma_1\gamma_2$
- If $\gamma_1\gamma_2 < 1 \implies$ the plot of $det(I - \gamma_2 G(j\omega))$ does not go through encircle the origin
- $\therefore$ By Nyq. criterion, $[I - \gamma_2 G(j\omega)]^{-1}$ is Hurwitz $\implies G_T$ and Z_T are Hurwitz

Examle Cont'd

- Now $Z_T(s)$ is SPR if

$$Z_T(j\omega) + Z_T^T(-j\omega) > 0 \quad \forall \omega \in R$$

$$\begin{aligned} & Z_T((j\omega) + Z_T^T(-j\omega) = [I + \gamma_2 G(j\omega)][I - \gamma_2 G(j\omega)]^{-1} \\ + \quad & [I - \gamma_2 G^T(-j\omega)]^{-1}[I + \gamma_2 G^T(-j\omega)] \\ = \quad & [I - \gamma_2 G^T(-j\omega)]^{-1}\left(2I - 2\gamma^2 G^T(-j\omega)G(j\omega)\right)[I - \gamma_2 G(j\omega)]^{-1} \end{aligned}$$

- Hence, $Z_T(j\omega) + Z_T^T(-j\omega)$ is **p.d.** iff

$$\sigma_{min}[I - \gamma_2^2 G^T(-j\omega)G(j\omega)] > 0 \quad \forall \omega \in R$$

$$\begin{aligned} \sigma_{min}[I - \gamma_2^2 G^T(-j\omega)G(j\omega)] & \geq 1 - \gamma_2^2 \sigma_{max}[G^T(-j\omega)]\sigma_{max}[G(j\omega)] \\ & \geq 1 - \gamma_1^2\gamma_2^2 > 0 \end{aligned}$$

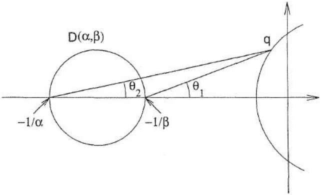
- Hence, we can conclude that the system is abs. stable if $\gamma_1\gamma_2 < 1$.

Circle Criterion

- This shows that closing the loop around a Hurwitz TF with a nonlinearity satisfying $\|\psi\| \leq \gamma_2\|y\|$ with sufficiently small γ_2 does not destroy the stability of the system (**small gain theorem**)
- In scalar case ($p = 1$), the conditions of the theorem can be verified graphically by Nyq. plot of $G(j\omega)$.
  - The conditions of the theorem are
    1.) $G_T(s) = \frac{G(s)}{1+\alpha G(s)}$ is Hurwitz
    2.) $Z_T(s) = \frac{1+\beta G(s)}{1+\alpha G(s)}$ is SPR
  - To verify SPR condition, $Z_T(s)$ is SPR iff $Z_T(s)$ is Hurwitz and $Re\left[\frac{1+\beta G(j\omega)}{1+\alpha G(j\omega)}\right] > 0 \ \forall \omega \ \in R$.
  - We now consider different cases:
  - First case $\beta > \alpha > 0$:
    - In this case, we have:

$$Re\left[\frac{1/\beta + G(j\omega)}{1/\alpha + G(j\omega)}\right] > 0 \ \forall \omega \ \in R$$

Circle Criterion



Graphical representation of the circle criterion.

- For a point q on the Nyq. plot of $G(j\omega)$, $1/\beta + G(j\omega)$ & $1/\alpha + G(j\omega)$ can be represented by the line connecting q to $-1/\beta + j0$ and $-1/\alpha + j0$
- The real part of the ratio of two complex numbers is positive when the angle difference is less than $\pi/2 \Longrightarrow \theta_1 - \theta_2 \ < \pi/2$ when q is outside $D(\alpha, \beta)$.
- $G_T(s)$ is Hurwitz, implying that $Z_T(s)$ is Hurwitz, if from the Nyq. criterion, the Nyq. plot of $G(j\omega)$ does not intersect the point $-1/\alpha + j0$ and encircle it exactly m times CCW where m is the $\sharp$ of poles of G in open RHP
- $\therefore$ Conditions of the theorem are satisfied if the Nyq. plot of $G(j\omega)$ does not enter the disk $D(\alpha, \beta)$ and encircle it m times CCW.

Circle Criterion

- **Second case:** $\beta > 0, \quad \alpha = 0$
  - The conditions of the theorem are: $G(s)$ is Hurwitz and $Re[1 + \beta G(j\omega)] > 0, \ \forall\omega \in R$

$$Re[G(j\omega)] > \frac{-1}{\beta}, \ \forall\omega \in R.$$

  - $\therefore$ Nyq. plot of $G(s)$ lies to the right of $s = \frac{-1}{\beta}$ line
- **Third case:** $\alpha < 0 < \beta$
  - The conditions of the theorem are:

$$Re\left[\frac{1/\beta + G(j\omega)}{1/\alpha + G(j\omega)}\right] < 0 \ \forall\ \omega \in R$$

  - $\therefore$ Nyq. plot of $G(s)$ must lie inside the disk $D(\alpha, \beta)$ (prove it).
    - Nyq. plot cannot encircle $\frac{-1}{\alpha} + j0 \implies G(s)$ must be Hurwitz for $G_T(s)$ to be so.

Circle Criterion

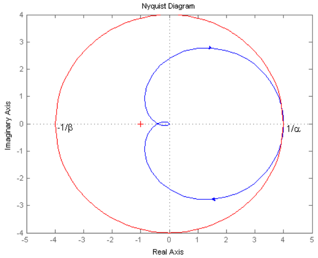
- **Theorem (Circle Criterion)**: *Consider system (9) with $p = 1$ and ψ satisfies the sector condition globally. The system is absolutely stable if one of the following conditions are satisfied:*
  1. *If $0 < \alpha < \beta$, the Nyq. plot of $G(j\omega)$ does not enter the disk $D(\alpha, \beta)$ and encircles it m times in CCW.*
  2. *If $0 = \alpha < \beta$, $G(s)$ is Hurwitz and the Nyq. plot of $G(j\omega)$ lies to the right of the line $s = \frac{-1}{\beta}$.*
  3. *If $\alpha < 0 < \beta$, $G(s)$ is Hurwitz and the Nyq. plot of $G(j\omega)$ lies inside $D(\alpha, \beta)$.*
- **Example 7.13:**

$$G(s) = \frac{4}{(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}$$

  - Since $G(s)$ is Hurwitz, we can allow α to be negative & apply the 3^{rd} case of the **circle criterion**.
  - We need $D(\alpha, \beta)$ that encloses the Nyq. plot.(This is not unique).
  - Select the disk $D(-\gamma_2, \gamma_2)$ where $\gamma_2 > 0$ to be chosen
  - If we set $\gamma_1 = \sup |G(j\omega)|$ and then use $\gamma_2\gamma_1 < 1$.

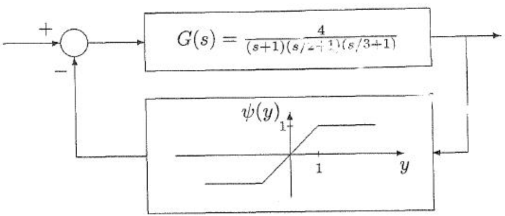
Example 7.13 Cont'd

- Maximum occurs at $\omega = 0$ and $\gamma_1 = 4 \implies \gamma_2 < 0.25$
- $\therefore$ System is absolutely stable $\forall$ nonlinearities in the sector $[-0.25 + \epsilon,\ 0.25 - \epsilon]$, where $\epsilon > 0$.
- By locating the center at another point, we may be able to obtain a disk that encloses the Nyq. plot more tightly.
- Let the center be at pt. $1.5 + j0$. The max distance to Nyq. plot is 2.834 $\rightsquigarrow$ radius could be 2.9 to ensure $D(-1/4.4, 1/1.4)$ encloses the Nyq. plot.
- System is absolutely stable $\forall$ nonlinearity in $[-.227,\ .714]$.



Example 7.13

- Alternatively, if we restrict $\alpha = 0$ and apply the 2^{nd} criterion, the Nyq. plot lies to the right of $s = -.857 \implies$ system is absolutely stable $\forall$ nonlinearity in $[0,\ 1.166]$.
- For example, using a nonlinearity like a saturation, we conclude that since ψ belongs to $[0,\ 1]$ sector $\implies$ system is **g.a.s.** using 2^{nd} condition.
- However, it fails the 3^{rd} criterion with $[-.25,\ .25]$ or $[-.227,\ .714]$ sectors.

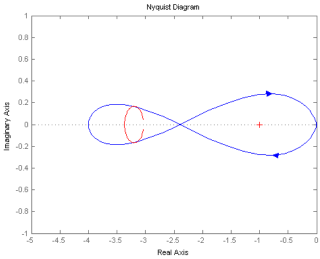


Feedback connection with saturation nonlinearity.

Example 7.14

$$G(s) = \frac{4}{(s-1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}$$

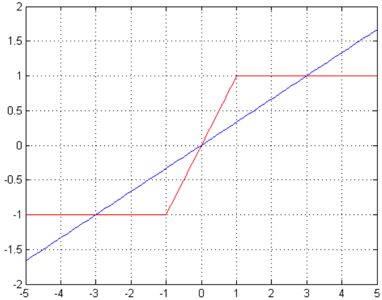
- $G(s)$ is **not** Hurwitz, we must restrict α to be positive & apply the 1^{st} case
  - The Nyq. plot must encircles the disk $D(\alpha, \beta)$ **once** in **CCW**.
  - A disk inside the right lobe is encircled once in CW $\longrightarrow$ no good.
  - A disk in the left lobe is encircled once in CCW.
  - Let us locate the center at $-3.2 + j0$ ($\approx$ half way between the two ends of the lobe).
  - Min. distance is .1688 $\implies$ choosing the radius .168 $\implies$ system is abs. stable $\forall$ nonlinearity in [.2969, .3298].



Example 7.15

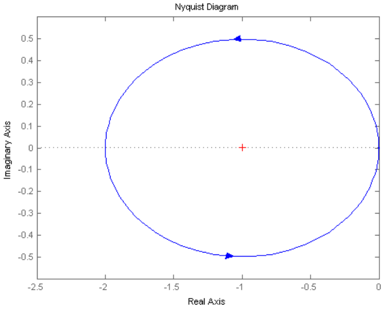
$$G(s) = \frac{s+2}{(s+1)(s-1)} \quad \text{and}$$
$$\psi(y) = sat(y)$$

- ψ belong to the sector $[0, \ 1]$.
- Since $G(s)$ is **not** Hurwitz, we must apply the 1^{st} case of the **circle criterion** which requires the sector cond. to hold with a **positive** α
- We cannot apply this theorem to $G(s)$ and conclude **abs. stability.**
- The best we can do is to show the **abs. stab. with a finite domain**



Example 7.15 Cont'd

- Now, ψ satisfies the sector condition with $\alpha = \frac{1}{a}$, & $\beta = 1$. on the interval $[-a, a]$.
- Nyq. plot of $G(s)$ must encircle the disk $D(\alpha, 1)$ once in **CCW**.
- Condition is satisfied with $\alpha > \ .5359$.
- Choosing $\alpha = .55 \implies$ sector interval is $[-1.818,\ 1.818]$ and the Disk $(.55, 1)$ is encircled once by the Nyq. plot in the CCW $\implies$ System is **abs. stable** with a finite domain.



- To estimate ROA, we use Lyapunove analysis. Let $V(x) = x^T P x$.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + u, \quad u = -\psi(y) \\ y &= 2x_1 + x_2\end{aligned}$$

- The loop transformation yields:

$$\begin{aligned} u &= \alpha y - \psi(y) = \psi_T(y) \\ \therefore \quad \dot{x} &= A_T x + B(-\psi_T(y)) \\ y &= Cx \\ A_T &= \begin{bmatrix} 0 & 1 \\ -.1 & -.55 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 2 & 1 \end{bmatrix}. \end{aligned}$$

- and ψ_T satisfies the sector condition with $K = 1 - \alpha = .45$.

- To find V, we need to find P, L, and C that satisfies KYP lemma

$$\therefore \; \epsilon = .02, \quad P = \begin{bmatrix} .4946 & .4834 \\ .4834 & 1.0774 \end{bmatrix}, \; L = \begin{bmatrix} .2946 & -.4436 \end{bmatrix}.$$

where ϵ is chosen s.t. $Z_T(s - .5\epsilon)$ is **PR** and $(\frac{\epsilon}{2} I + A_T)$ is Hurwitz.

Example 7.15 Cont'd

- $\therefore \; V = x^T P x$ is a Lyap. fcn for the system and

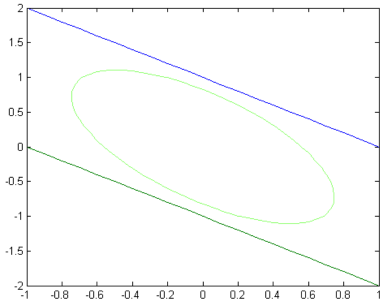
$$\Omega_c = \{x \in R^2 \mid V(x) \leq c\}$$

where

$$c \leq \min_{\{|y|=1.818\}} V(x) = .3446$$

to ensure that Ω_c is contained in the set $\{|y| \leq 1.818\}$.

- $\therefore$ taking $c = .34$ gives the estimate of ROA.



Popov Criterion

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -\psi(t, y) \end{aligned} \tag{9}$$

where $x \in R^n$, $u, y \in R^P$, (A, B) is controllable, (A, C) is observable, and $\psi : R^P \longrightarrow R^P \rightsquigarrow \psi^T(\psi - Ky) \leq 0$ with $K = diag(\beta_1, ..., \beta_p)$ in $\Gamma = \{y \in R^p \mid a_i \leq y_i \leq b_i\}$.

- We chose a Lure-type Lyapunov function

$$\begin{aligned} V(x) &= x^T Px + 2\eta \int_0^y \psi(\sigma) K d\sigma, \quad P = P^T > 0, \ \eta \geq 0 \text{ to be chosen} \\ \therefore \dot{V} &= x^T(PA + A^T P)x - 2x^T PB\psi(y) + 2\eta\, \psi^T(y)KC[Ax - B\psi(y)] \\ \dot{V} &\leq x^T(PA + A^T P)x - 2x^T PB\psi(y) + 2\eta\psi^T(y)KC[Ax - B\psi(y)] \\ &- 2\psi^T(y)(\psi(y) - Ky) \\ &= x^T(PA + A^T P)x - 2x^T(PB - C^T K - \eta A^T C^T K)\psi \\ &- \psi^T(y)(2I + \eta KCB + \eta B^T C^T K)\psi(y) \end{aligned}$$

Popov Criterion

- Chose η s.t. $2I + \eta KCB + \eta B^T C^T K \geq 0$ by setting η sufficiently small.
- Now, let $2I + \eta KCB + \eta B^T C^T K = W^T W$ and suppose $\exists P = P^T > 0$ and L and $\epsilon > 0$ s.t.

$$\begin{aligned} PA + A^T P &= -L^T L - \epsilon P \\ PB &= C^T K + \eta A^T C^T K - L^T W \end{aligned}$$

- Then
$$\begin{aligned} \dot{V} &\leq -\epsilon x^T P x - [Lx - W\psi(y)]^T [Lx - W\psi(y)]^T \\ &\leq -\epsilon x^T P x \; < 0 \end{aligned}$$

- Existence of L, P, and ϵ can be verified from KYP lemma iff
$$\begin{aligned} Z(s) &= I + \eta KCB + (KC + \eta KCA)(sI - A)^{-1} B \\ &= I + (1 + \eta s) KG(s), \text{ is } \textbf{SPR.} \text{ where } Z(\infty) + Z^T(\infty) = W^T W \end{aligned}$$

- Suppose η is chosen s.t. $(1 + \eta\lambda_i) \neq 0$ for all eig. of A, i.e. $-1/\eta$ is not an eig. of $A \rightsquigarrow (A, KC + \eta KCA)$ is observable since (A, c) is observable.

Popov Criterion

- **Multivariable Popov Criterion:** *Consider the system where A is Hurwitz (A, B) is controllable, (C, A) is observable and ψ is time-invariant decentralized nonlinearity satisfying the sector condition globally with a diagonal matrix K. The, the system is* **absolutely stable** *if $\exists \eta \geq 0$ with $-1/\eta$ not an eig. of A, s.t.*

$$Z(s) = I + (1 + \eta s)KG(s)$$

is **SPR**.

  - loop transformation can be applied for Popov criterion as well
  - when $p = 1 \implies$ choose η s.t. $Z(\infty) > 0$. $Z(s)$ is SPR iff

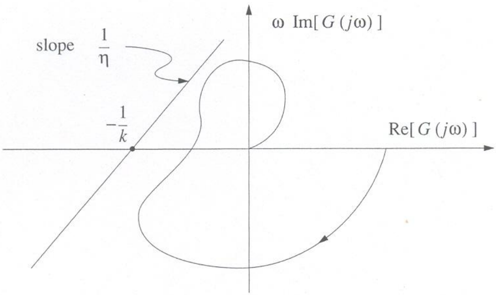
$$Re[1 + (1 + j\eta\omega)kG(j\omega)] > 0 \ \forall \omega \in R$$

$$\frac{1}{k} + Re[G(j\omega)] - \eta\omega Im[G(j\omega)] > 0 \ \forall \omega \in R \qquad (10)$$

that is the plot of $Re[G(j\omega)]$ vs. $\omega Im[G(j\omega)]$ lies to the right of the line crossing the point $\frac{-1}{k} + j0$ with the slope $\frac{1}{\eta}$.

Popov Criterion

- The plot is known as Popov plot.
- For $\eta = 0$ condition (10) reduces to circle criterion
  - $\therefore$ Popov criterion is weaker than circle criterion



Example 7.16

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_2 - h(y)\\ y &= x_1\end{aligned}$$

- $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ is not Hurwitz $\Longrightarrow$ use a loop transformation:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_2 - \alpha y + \alpha y - h(y)\\ y &= x_1\end{aligned}$$

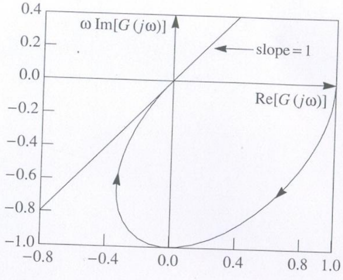
- Now, take $A = \begin{bmatrix} 0 & 1 \\ -\alpha & -1 \end{bmatrix}$,

  $\psi(y) = h(y) - \alpha y,\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

- Assume h belongs to sector $[\alpha,\ \beta]$, $\beta > \alpha \implies \psi$ belongs to the sector $[0,\ K]$, $K = \beta - \alpha$

Example 7.15 cont'd

- $Z(s) = 1 + (1 + \eta s)KG(s), \quad G(s) = C(sI - A)^{-1}B, \quad \therefore \; Z(\infty) = 1 \; \forall \eta.$
- We have $\frac{1}{k} + \frac{\alpha - \omega^2 + \eta\omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \quad \forall \; \omega \; \in \; R$
- Since $k > 0 \implies$ for $\eta \; \geq 1$ the inequality is satisfied.
- Let $\alpha = 1 \implies \frac{1}{k} + \frac{1 - \omega^2 + \eta\omega^2}{(1 - \omega^2)^2 + \omega^2} > 0$
- Popov plot lies to the right of any line of slope $\leq \; 1$.
- $\therefore$ System is abs. stable $\forall \; h(.)$ in the sector $[\alpha, \; \beta]$ with α sufficiently small and β sufficiently large.



Example 7.15 cont'd

- To check the advantage of Popov criterion: $\eta = 0$
- the system is abs. stable if Nyquist plot of $G(j\omega)$ lies to right of vertical line $Re[s] = -1/k$
  - $\therefore$ k cannot be arbitrarily large
  - $\frac{1}{k} + \frac{\alpha - \omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0 \rightsquigarrow k < 1 + 2\sqrt{\alpha}$
- $\therefore$ by circle criterion the system is abs stable when $h(.)$ is in the sector $[\alpha, 1 + +2\sqrt{\alpha} - \epsilon]$ where $\alpha > 0$ and $\epsilon > 0$ is arbitrarily small