Nonlinear Control
Lecture 7: Passivity

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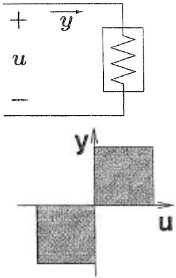
Passivity Definition

- Consider a memoryless function

$$y = h(t, u) \tag{1}$$

where
  - $h : [0, \infty) \times R^p \to R^p$
  - u: input; y: output
- Exp. Resistive element: u is voltage; y is current
  - It is passive if the inflow of power is always nonneg.
  - $\therefore$ $uy \geq 0$ for all (u, y)
  - Geometrically it means the $u - y$ curves lie in first and third quadrant
  - The simplest option is linear resistor ($u = Ry$)



- If u and y are vectors, the power flow onto the network will be

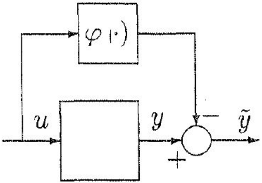
$$u^T y = \sum_{i=1}^{p} u_i y_i = \sum_{i=1}^{p} u_i h_i(u)$$

- For time-varying system, as long as the passivity condition is satisfied for all time, it is called passive.
- Extreme case of passivity : $u^T y = 0 \rightsquigarrow$, this system is **lossless**.
- **Input strictly passivity**: if a fcn. h satisfies $u^T y \geq u^T \phi(u)$, and $u^T \phi(u) > 0, \forall u \neq 0$
  - since $u^T y = 0$ only if $u = 0$

Input Feedforward Passive

- Let us define a new output: $\tilde{y} = y - \phi(u)$:

$$u^T\tilde{y} = u^T[y - \phi(u)] \geq u^T\phi(u) - u^T\phi(u) =$$

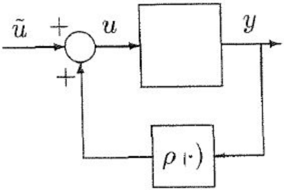


- $\therefore$ any fcs. satisfying $u^Ty \geq u^T\phi(u)$ can be transformed into a passive fcn. via input feedforward.
- This fcs is input feedforward passive

Output Feedback Passive

- Suppose $u^T y \geq y^T \rho(y)$.
- Let us define a new input: $\tilde{u} = u - \rho(y)$:

$$\tilde{u}^T y = [u - \rho(y)]^T y \quad \geq \quad y^T \rho(y) - y^T \rho(y) = 0$$



- $\therefore$ any fcs. satisfying $u^T y \geq y^T \rho(y)$ can be transformed into a passive fcn. via output feedback.
- This fcs is output feedback passive
- If $y^T \rho(y) > 0, \forall y \neq 0$ the fcn. is called output strictly passive
  - since $u^T y = 0$ only if $y = 0$

Passivity of Memoryless Fcn.

- The system $y = h(t, u)$ is
  - passive if $u^T y \geq 0$
  - lossless if $u^T y = 0$
  - input-feedforward passive if $u^T y \geq u^T \phi(u)$ for some fcn ϕ
  - input strictly passive if $u^T y \geq u^T \phi(u)$ and $u^T \phi(u) > 0, \forall u \neq 0$
  - output-feedback passive if $u^T y \geq y^T \rho(y)$ for some fcn ρ
  - output strictly passive if $u^T y \geq y^T \rho(y)$ and $y^T \rho(y) > 0, \forall y \neq 0$

State Model

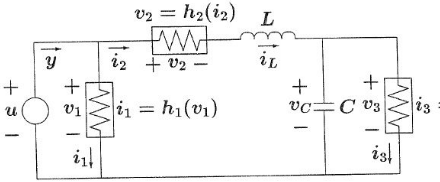
- Consider a dynamical system with state model

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{2}$$

  - $f : R^n \times R^p \to R^n$ is local lip.
  - $h : R^n \times R^p \to R^p$ is cont.
  - $f(0,0) = 0$ and $h(0,0) = 0$
  - $\#$ inputs $= \#$ outputs

Motivated Example: RLC Circuit

- Consider the RLC circuit with linear C and L and nonlinear R
- The nonlinear resistors are represented by: $i_1 = h_1(v_1)$; $v_2 = h_2(i_2)$; $i_3 = h_3(v_3)$
- Input u: voltage; output y: current
- power flow into the network: uy
- Define x_1: current through L; x_2: voltage across C



- $\therefore$ state model

$$\begin{aligned} L\dot{x}_1 &= u - h_2(x_1) - x_2 \\ C\dot{x}_2 &= x_1 - h_3(x_2) \\ y &= x_1 + h_1(u) \end{aligned}$$

- The system is passive if the absorbed energy by the network is greater than the stored energy in the network over the same period:

$$\int_0^t u(s)y(s)ds \geq V(x(t)) - V(x(0)) \tag{3}$$

  where $V(x) = 1/2Lx_1^2 + 1/2Cx_2^2$: stored energy
- Strict inequality of (3) yields difference between the absorbed energy and increased stored energy equals to dissipative energy in the resistors
- (3) hold for every $t \geq 0 \rightsquigarrow$ for all t $u(t)y(t) \geq \dot{V}(x(t), u(t))$
- i.e. the power flow must be greater than or equal to the rate of change of the stored energy

Motivated Exp. Cont'd

- Take the derivative of V along the system traj: $\dot{V} = uy - uh_1(u) - x_1 h_2(x_1) - x_2 h_3(x_2)$
- $\therefore uy = \dot{V} + uh_1(u) + x_1 h_2(x_1) + x_2 h_3(x_2)$
- If h_1, h_2 and h_3 are passive $\Rightarrow uy \geq \dot{V}$; system is passive
- Otherwise
  - Case 1: If $h_1 = h_2 = h_3 = 0, uy = \dot{V} \rightsquigarrow$ no energy dissipation, system is lossless
  - Case 2: If $h_2, h_3 \in$ sector $[0, \infty]$ (passive fcns.) $\rightsquigarrow uy \geq \dot{V} + uh_1(u)$
    - If $uh_1(u) > 0$ for all $u \neq 0 \rightsquigarrow$ it is input strick passive (absorbed energy is greater than increased stored energy unless $u(t) \equiv 0$)
    - If $uh_1(u) < 0$ for some u it can be made passive by an input ff
  - Case 3: If $h_1 = 0$ and h_3: passive fcn ($\in$ sector $[0, \infty]$) $\rightsquigarrow uy \geq \dot{V} + yh_2(y)$
    - If $yh_2(y) > 0$ for all $y \neq 0 \rightsquigarrow$ it is output strick passive (absorbed energy is greater than increased stored energy unless $y(t) \equiv 0$)
    - If $yh_2(y) < 0$ for some u it can be made passive by an output fb

Motivated Exp. Cont'd

- Case 4: If $h_1 \in [0, \infty]$ and $h_2, h_3 \in (0, \infty)$
  $\rightsquigarrow uy \geq \dot{V} + x_1 h_2(x_1) + x_2 h_3(x_2)$
  - where $x_1 h_2(x_1) + x_2 h_3(x_2)$ is pos. def.
  - It is state strict passive or simply strick passive (absorbed energy is greater than increased stored energy unless $x(t) \equiv 0$)

Passivity Based on State Model

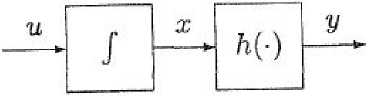
- The system (2) is passive if there exist a cont. diff. p.s.d fcn $V(x)$ (called storage fcn) s.t.

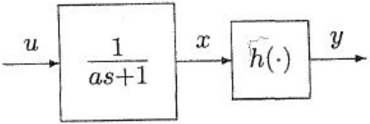
$$u^T y \quad \geq \quad \dot{V} = \frac{\partial V}{\partial x} f(x,u), \quad \forall (x,u) \in R^n \times R^n$$

  Moreover, it is
  - Lossless if $u^T y = \dot{V}$
  - Input-feedforward passive if $u^T y \geq \dot{V} + u^T \phi(u)$ for some fcn ϕ
  - Input strictly passive if $u^T y \geq \dot{V} + u^T \phi(u)$ and $u^T \phi(u) > 0, \forall u \neq 0$
  - Output-feedback passive if $u^T y \geq \dot{V} + y^T \rho(y)$ for some fcn ρ
  - Output strictly passive if $u^T y \geq \dot{V} + y^T \rho(y)$ and $y^T \rho(y) > 0, \forall y \neq 0$
  - Strictly passive if $u^T y \geq \dot{V} + \psi(x)$ for some p.d. ψ
- In all cases, the inequality should hold for all (x, u)

Example

- Consider a cascade connection of an integrator and a passive memoryless fcn. $\dot{x} = u, \; y = h(x)$
- h is passive $\rightsquigarrow \int_0^x h(\sigma)d\sigma \geq 0, \quad \forall x$
- Storage fcn: $V(x) = \int_0^x h(\sigma)d\sigma$
- $\therefore \dot{V} = h(x)\dot{x} = yu$ it is loss less
- Now replace the integrator with $1/(as+1)$, $a > 0$
- The state model is: $a\dot{x} = -x + u, \; y = h(x)$
- $V = a\int_0^x h(\sigma)d\sigma \rightsquigarrow \dot{V} = h(x)(-x+u) = yu - xh(x) \leq yu$
- $\therefore$ It is passive.
- When $xh(x) > 0$ it is strictly passive





$\mathcal{L}_2$ and Lyapunov Stability

- **Lemma:** *If the system (2) is output strictly passive with $u^T y \geq \dot{V} + \delta y^T y$ for some $\delta > 0$ then it is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain less than or equal to $1/\delta$*
- **Definition:** *The system (2) is zero-state observable if no solution of $\dot{x} = f(x, 0)$ can stay identically in $S = \{x \in R^n | h(x, 0) = 0\}$ other than the trivial solution $x(t) \equiv 0$*
- **Example:** For linear system $\dot{x} = Ax, \quad y = Cx$
  - Observability is equivalent to $y(t) = Ce^{At}x(0) \equiv 0 \Leftrightarrow \begin{cases} x(0) = 0 \\ x(t) \equiv 0 \end{cases}$

$\mathcal{L}_2$ and Lyapunov Stability

- **Lemma:** *If system (2), is passive with a p.d. storage fcn. $V(x)$, then the origin of $\dot{x} = f(x, 0)$ is stable.*
- **Lemma:** *For system (2), the origin of $\dot{x} = f(x, 0)$ is a.s if the system is*
  - *strictly passive or*
  - *output strictly passive and zero-state observable*
- *Furthermore, if the storage fcn. is radially unbounded, then the origin will be g.a.s*

Example

- Consider SISO system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_1^3 - kx_2 + u \\ y &= x_2 \end{aligned}$$

- $a > 0, \; k > 0$
- Consider p.d. radially unbounded $V(x) = (1/4)ax_1^4 + (1/2)x_2^2$
- $\dot{V} = -ky^2 + yu$
- By $\rho(y) = ky$, it is output strictly passive $\rightsquigarrow$ $\mathcal{L}_2$ f.g.s. with gain less than or equal to $1/k$
- When $u = 0$, $y(t) \equiv 0 \Rightarrow x_2 \equiv 0 \Rightarrow x_1 \equiv 0 \rightsquigarrow$ zero-state observable
- $\therefore$ it is g.a.s

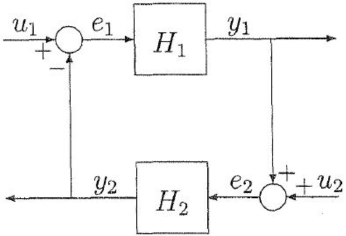
Feedback and Passivity Theorems

- Consider a fb connection of H_1 and H_2
  - in time-invariant dynamical system represented by state model

$$\begin{aligned} \dot{x}_i &= f_i(x_i, e_i) \\ y_i &= h_i(x_i, e_i) \end{aligned} \quad (4)$$

  - or (possibly time-varying) memoryless fcn

$$y_i = h_i(t, e_i) \quad (5)$$



- Objective: Analyze stability of fb connection, using passivity properties of fb components (H_1 and H_2)

Feedback and Passivity Theorems

1. If both components H_1 and H_2 are dynamical systems, the closed-loop state model is

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \quad (6)$$

- where $x = [x_1 \;\; x_2]^T, \;\; u = [u_1 \;\; u_2]^T, \;\; y = [y_1 \;\; y_2]^T$
- Assuming $f_i(0,0) = 0$ and $h_i(0,0) = 0 \rightsquigarrow f(0,0) = 0$ and $h(0,0) = 0$
- Assuming f_i and h_i are locally lip. $\rightsquigarrow$ f and h are locally lip.

Feedback and Passivity Theorems

2. If H_1 is a dynamical system and H_2 is memoryless fcn., the closed loop state model is

$$\begin{aligned} \dot{x} &= f(t, x, u) \\ y &= h(t, x, u) \end{aligned} \quad (7)$$

   - where $x = x_1, \; u = [u_1 \; u_2]^T, \; y = [y_1 \; y_2]^T$
   - Assume $f(t, 0, 0) = 0$ and $h(t, 0, 0) = 0$
   - Assume f is p.c. in t and locally lip. in (x, u), and h is p.c. in t and cont. in (x, u)

3. If both components are memoryless fcns
   - It can be considered as a special case when x does not exit

Feedback and $\mathcal{L}_2$ Stability

- **Theorem:** *The feedback connection of two passive system is passive*
  - **Proof:** Let $V_1(x_1)$ and $V_2(x_2)$ are storage fcns of H_1 and H_2 respectively.
  - If either components are memoryless fcn, take $V_i = 0$
  - Then $e_i^T y_i \geq \dot{V}_i$
  - Considering fb. connection
    $e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$
  - $\therefore u^T y = u_1^T y_1 + u_2^T y_2 \geq \dot{V}_1 + \dot{V}_2 = \dot{V}$
- **Lemma:** *The fb connection of two output strictly passive systems with*

$$e_i^T y_i \quad \geq \quad \dot{V}_i + \delta_i y_i^T y_i + \epsilon_i e_i^T e_i, \;\; i = 1, 2$$

is finite-gain $\mathcal{L}_2$ stable with gain if $\epsilon_1 + \delta_2 > 0, \;\; \epsilon_2 + \delta_1 > 0$

Example

- Consider $H_1 : \begin{cases} \dot{x} = f(x) + G(x)e_1 \\ y_1 = h(x) \end{cases}$ and $H_2 : y_2 = ke_2$ where $k > 0$, $e_i, y_i \in \mathcal{R}^p$
- Suppose there is a p.d. fcn $V_1(x)$ s.t. $\frac{\partial V_1}{\partial x} f(x) \leq 0,\ \frac{\partial V_1}{\partial x} G(x) = h^T(x),\ \forall x \in \mathcal{R}^n$
- Both components are passive
- and $e_2^T y_2 = ke_2^T e_2 = \gamma ke_2^T e_2 + \frac{(1-\gamma)}{k} y_2^T y_2,\ 0 < \gamma < 1$
- $\therefore\ \epsilon_1 = \delta_1 = 0, \epsilon_2 = \gamma k,\ \delta_2 = \frac{(1-\gamma)}{k}$
- It is finite gain $\mathcal{L}_2$ stable

Feedback and A.S.

- Stability of origin is trivial if both components are passive. (Tell me why?.
- Let us focus on a.s.
- **Theorem:** *Consider fb connection of two T.I. dynamical systems of the form (4). The origin of the closed-loop system (6) (when $u = 0$) is a.s. if*
  - *both components are strictly passive*
  - *or both components are output strictly passive and zero-state observable*
  - *or one component is strictly passive and the other one is output strictly passive and zero-state observable*
- *Furthermore, if storage fcn of each component is radially unbounded, the origin is g.a.s*

Example

- Consider fb connection with:

$$H_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_1^3 - kx_2 + e_1 \\ y_1 = x_2 \end{cases}, \quad H_2 : \begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = -bx_3 - x_4^3 + e_2 \\ y_2 = x_4 \end{cases}$$

$a, b, k > 0$

- Use $V_1 = (a/4)x_1^4 + (1/2)x_2^2 \rightsquigarrow \dot{V}_1 = -ky_1^2 + y_1 e_1$
- $\therefore$ H_1 is output strictly passive.
- when $e_1 = 0$, $y_1 \equiv 0 \Leftrightarrow x_2 \equiv 0 \Leftrightarrow x_1 \equiv 0$
- $\therefore$ H_1 is zero-state observable
- Use $V_2 = (b/2)x_3^2 + (1/2)x_4^2 \rightsquigarrow \dot{V}_2 = -y_2^4 + y_2 e_2$
- $\therefore$ H_2 is output strictly passive.
- when $e_2 = 0$, $y_2 \equiv 0 \Leftrightarrow x_4 \equiv 0 \Leftrightarrow x_3 \equiv 0$
- $\therefore$ H_2 is zero-state observable
- V_1, V_2 are radially unbounded $\rightsquigarrow$ the closed-loop system is g.a.s

Example

- Reconsider the pervious system but change the output of H_1 to $y_1 = x_2 + e_1$
- Hence $\dot{V}_1 = -k(y_1 - e_1)^2 - e_1^2 + y_1 e_1 \rightsquigarrow H_1$ is passive **Not strictly passive or output strictly passive**
- Consider Lyap fcn of closed-loop sys. $V = V_1 + V_2 = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2 + \frac{1}{2}bx_3^2 + \frac{1}{2}x_4^2$
- $\dot{V} = -kx_2^2 - x_4^4 - x_4^2 \leq 0$
- Also, $\dot{V} = 0 \Rightarrow x_2 = x_4 = 0$
  $x_2 \equiv 0 \Rightarrow x_1 \equiv 0$
  $x_4 \equiv 0 \Rightarrow x_3 \equiv 0$
- V is radially unbounded $\rightsquigarrow$ the closed-loop system is g.a.s

Feedback and A.S.

- **Theorem:** *Consider fb connection of a strictly passive, T.I. dynamical systems of the form (4) and a passive (possibly time-varying) memoryless fcn of the form (5). The origin of the closed-loop system (7) (when $u = 0$) is u.a.s. Furthermore, if storage fcn of the dynamical system is radially unbounded, the origin is g.u.a.s*