Nonlinear Control
Lecture 6: Stability Analysis III

Farzaneh Abdollahi

Department of Electrical Engineering

Amirkabir University of Technology

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- Stability of Perturbed Systems
  - Vanishing Perturbation
  - Nonvanishing Perturbation

Stability of Perturbed Systems

- Consider the system: $$\dot{x} = f(t,x) + g(t,x) \quad (1)$$

  where $f : [0,\infty) \times D \longrightarrow R^n$, $g : [0,\infty) \times D \longrightarrow R^n$ are p.c. in t and locally Lip. in x on $[0,\infty) \times D$, $D \in R^n$.
  - The system can be regarded as a perturbation of:

    $$\dot{x} = f(t,x) \quad (2)$$

  - The perturbation could result from modeling error, aging, or uncertainties and disturbances.
  - Generally, $g(t,x)$ is not exactly known, rather some info. like an upper bound is known
  - Uncertainties that do not change the order of the system can always be represented in additive form
  - Now suppose the origin of nominal system (2) is **u.a.s.**
  - what can be said about the stability of the perturbed system (1)?
  - Use the Lyap. fcn of the nominal system for perturbed system

Vanishing Perturbation

- The conclusion depends on whether $g(t,x)$ is vanishing at the origin.
- If $g(t,0)=0$, the perturbed system has an Equ. pt. at the origin $\Longrightarrow$ we study the stability of the Equ. pt. (origin)
- If $g(t,0)\neq 0$, the origin will not be an Equ. pt. $\Longrightarrow$ we study ultimate boundedness of the solution
- **Vanishing Perturbation**
  - Let $g(t,0)=0$, so $x=0$ is an Equ. pt. of the perturbed as well as the nominal system.
  - Let $V(t,x)$ is a Lyap. fcn satisfying:

$$\begin{aligned} c_1\,\|x\|^2 \;&\le\; V(t,x) \;\le\; c_2\,\|x\|^2 \\ \frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x) \;&\le\; -c_3\|x\|^2 \\ \frac{\partial V}{\partial x} \;&\le\; c_4\|x\| \qquad\qquad \forall (t,x)\in[0,\infty)\times D, \end{aligned}$$

for some pos. const. $c_1,\ldots,c_4$ for the nominal system.

Vanishing Perturbation

- Let the perturbation satisfies the linear growth bound:

$$\|g(t,x)\| \; \leq \; \gamma(t) \; \|x\| \;\; \forall t \; \geq \; 0, \;\; \forall \; x \; \in \; D,$$

  where $\gamma : \; R \; \longrightarrow \; R$ is nonnegative and p.c. $t \; \geq \; 0.$

- Use $V(t,x)$ to study the stability of $x = 0$ as an Equ. pt. for the perturbed system. We have:

$$\begin{aligned} \dot{V}(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x} g(t,x) \\ \dot{V}(t,x) &\leq -c_3\|x\|^2 + \|\frac{\partial V}{\partial x}\| \|g(t,x)\| \\ &\leq -c_3\|x\|^2 + c_4\gamma(t)\|x\|^2 \end{aligned}$$

- If $\gamma(t)$ is small enough to satisfy the bound

$$\gamma(t) \; \leq \; \bar{\gamma} \; \leq \; c_3/c_4 \quad \forall t \; \geq \; 0 \tag{3}$$

  then $\dot{V}(t,x) \; \leq \; -(c_3 - \bar{\gamma}c_4)\|x\|^2 \leq 0, \;\; (c_3 - \bar{\gamma}c_4) \; > 0$

Vanishing Perturbation

- **Theorem:** *Let* $x = 0$ *be an* **e.s.** *Equ. pt. of the nominal system. Let* $V(t, x)$ *be a Lyap. fcn of the nominal system satisfying the above 3 inequalities in* $[0, \infty) \times D$, $D = \{x \in R^n | \; \|x\| < r\}$. *Suppose the perturbation term satisfies the growth condition. Then* $x = 0$ *is an* **e.s.** *Equ. pt. of the perturbed system.*
- *Moreover, if the assumption holds globally, then the origin is* **g.e.s.**
- $\therefore$ **e.s.** is *robust* w.r.t. a class of perturbation (note that we do not need to know $V(t, x)$ explicitly.

Example 6.1 for Vanishing Perturbation

- Consider
$$\dot{x} = Ax + g(t, x)$$
- where A is Hurwitz and $\|g(t, x)\| \leq \bar{\gamma}\ \|x\|\ \forall t \geq 0$
- Let $Q = Q^T > 0$ and solve Lyap. Equ. $A^T P + PA = -Q$ for P.
- A is Hurwitz $\Longrightarrow \exists P = P^T > 0.$ and $V(x) = x^T Px$ satisfying the 3 inequalities, i.e.
$$\lambda_{min}(P)\|x\|^2 \leq V(x) \leq \lambda_{max}(P)\|x\|^2$$
$$\frac{\partial V}{\partial x} Ax = -x^T Qx \leq -\lambda_{min}(Q)$$
$$\|\frac{\partial V}{\partial x}\| = \|2x^T P\| \leq 2\lambda_{max}(P)\|x\|$$
- Then $\dot{V}$ along the trajectories of perturbed system:
$$\dot{V} \leq -\lambda_{min}(Q)\ \|x\|^2 + 2\lambda_{max}(P)\bar{\gamma}\|x\|^2$$
- $\therefore x = 0$ is **e.s.** if $\bar{\gamma} < \lambda_{min}(Q)/2\lambda_{max}(P)$

Example 6.2 for Vanishing Perturbation

- Consider
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -4x_1 - 2x_2 + \beta x_2^3, \quad \beta \geq 0 \textbf{ unknown}\end{aligned}$$

  - Define $f(x) = Ax = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $g(x) = \begin{bmatrix} 0 \\ \beta x_2^3 \end{bmatrix}$
  - $\lambda\{A\} = -1 \pm j\sqrt{3} \implies A$ is Hurwitz
  - $\therefore\ A^T P + PA = -I \implies P = \begin{bmatrix} 3/2 & 1/8 \\ 1/8 & 5/16 \end{bmatrix}$
  - $\therefore\ V(x) = x^T P x \implies c_3 = 1,\ c_4 = 2\lambda_{max}(P) = 3.026$
  - $g(x)$ satisfies $\|g(x)\| = \beta|x_2^3| \leq \beta k_2^2|x_2| \leq \beta k_2^2\|x\|\ \forall |x_2| \leq k_2$.
  - Hence, $\dot{V}(x) \leq -\|x\|^2 + 3.06\ \beta k_2^2\|x\|^2$
  - $\dot{V}$ is n.d. if $\beta < \frac{1}{3.026\ k_2^2}$.
  - To estimate the bound k_2, let $\Omega_c = \{x \in R^2 |\ V(x) \leq c\}$.

Example 6.2 for Vanishing Perturbation

- The boundary of Ω_c is the Lyap. surface:
$$V(x) = \frac{3}{2}x_1^2 + \frac{1}{4}x_1x_2 + \frac{5}{16}x_2^2 = c$$
- The largest value of $|x_2|$ can be obtained by differentiating w.r.t. $x_1 \implies 3x_1 + \frac{1}{4}x_2 = 0$.
- The extreme values of x_2 are obtained by intersecting $x_1 = \frac{-x_2}{12}$ with $V = c$.
- The largest value of x_2^2 in $V = c$ is $\frac{96}{29}c$. Thus all pts. inside Ω_c satisfy the bound $|x_2| \leq k_2$ where $k_2^2 = \frac{96c}{29}$.
- if $\beta < \frac{29}{3.026\times 96c} \approx \frac{0.1}{c} \implies \dot{V}$ is n.d. in Ω_c and $x = 0$ is e.s with Ω_c as an estimate of the RoA.
- The above inequality shows a trade of between the upper bound of β and the estimate of the region of attraction.
- The smaller the upper bound, the larger the RoA.

Vanishing Perturbation

- The results above can be extended to **a.s.**
- However, the growth bound on perturbation would depend on the nature of the Lyap. fcn of the nominal system.

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x) \ \forall\, (t,x) \in [0,\infty) \times D,$$

where W_3 is p.d. and cont.

- Therefore for perturbed system:

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x} g(t,x) \leq -W_3(x) + \|\frac{\partial V}{\partial x} g(t,x)\|$$

- Our task is to show $\|\frac{\partial V}{\partial x} g(t,x)\| < W_3(x)$

Vanishing Perturbation

- One class of Lyap. fcn for which the analysis is simple when $V(t,x)$ is **p.d., decresent**, and satisfies:

$$\begin{aligned}\dot{V}(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3 \phi^2(x) \\ \left\| \frac{\partial V}{\partial x} \right\| &\leq c_4 \phi(x) \ \forall \, (t,x) \in [0,\infty) \times D,\end{aligned}$$

  where $\phi : \ R^n \longrightarrow R$ is p.d. and cont.
- A lyap. fcn satisfying above conditions is called quadratic-type Lyap. fcn.
- Then for perturbed system:

$$\begin{aligned}\dot{V}(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x} g(t,x) \leq -c_3 \phi^2(x) \\ &+ c_4 \phi(x) \| g(t,x) \|\end{aligned}$$

- Suppose $\|g(t,x)\| \leq \gamma \, \phi(x), \quad \gamma < c3/c4$, then

  $\dot{V}(t,x) \leq -(c_3 - c_4\gamma)\phi^2(x) \rightsquigarrow \dot{V}$is **n.d.** $\Longrightarrow$ the system is **g.a.s.**

Example 6.3 for Vanishing Perturbation

$$\dot{x} = -x^3 + g(t, x)$$

- The nominal system $\dot{x} = -x^3$ has a **g.a.s.** Equ. pt. at $x = 0$.
- However, it is not **e.s.** since no Lyap. fcn. exist that satisfies the 3 inequalities for linearized model.
- Let $V(x) = x^4$, the above conditions are satisfied with $\phi(x) = |x|^3, \ c_3 = 4, c_4 = 4$.
- Let $|g(t, x)| \ \leq \ \gamma|x|^3, \ \forall \ x$ with $\gamma < 1 \implies \dot{V} \ \leq \ -4(1 - \gamma)\phi^2$
- $\therefore \ x = 0$ is **g.a.s.**

Nonvanishing Perturbation

- The general case when $g(0, t) \neq 0$ shall be treated differently since $x = 0$ is no longer an Eq. pt.
- The best we can hope is to expect $x(t)$ is ultimately bounded with a small bound if $g(t, x)$ is small in some sense.
- Start with the case when the origin of the nominal system is **e.s.**
- **Lemma:** *Let $x = 0$ be an* **e.s** *Eq. pt. of $\dot{x} = f(t, x)$ (***nominal system***). Let $V(t, x)$ be a Lyap. fcn. of the* **nominal system** *satisfying the 3 inequalities in $[0, \infty) \times D$, where $D = \{x \in \mathcal{R}^n | \|x\| < r\}$. Let the perturbation term satisfies*

$$\|g(t, x)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r \quad \forall t \geq 0, \quad \forall x \in D, \quad 0 < \theta < 1$$

then $\forall \|x(t_0)\| < \sqrt{\frac{c_1}{c_2 r}}$, the sol. of the perturbed system satisfies

Example for Nonvanishing Perturbation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4x_1 - 2x_2 + \beta x_2^3 + d(t), \quad \beta \geq 0 \end{aligned}$$

where $\beta \geq 0$ is unknown & $d(t)$ is unif. bounded disturbance s.t. $|d(t)| \leq \delta \; \forall \; t \geq 0$.

- $V = x^T P x$ with $P = \begin{bmatrix} 3/2 & 1/8 \\ 1/8 & 5/16 \end{bmatrix}$ is a Lyap. fcn for nominal system.
- βx_2^3 is vanishing and $d(t)$ is nonvanishing perturbation.

$$\begin{aligned} \dot{V} &= -\|x\|_2^2 + 2\beta x_2^2 \left(\frac{1}{8} x_1 x_2 + \frac{5}{16} x_2^2 \right) + 2d(t) \left(\frac{1}{8} x_1 + \frac{5}{16} x_2 \right) \\ &\leq -\|x\|_2^2 + \frac{3}{4} \beta k_2^2 \|x\|_2^2 + \frac{\sqrt{29}\delta}{8} \|x\|_2 \end{aligned}$$

where the inequality $|2x_1 + 5x_2| \leq \|x\| \sqrt{4+25}$ & $|x_2| \leq k_2$.

Example for Nonvanishing Perturbation

- Suppose $\beta \leq 4(1-\zeta)/3k_2^2, \quad 0 < \zeta < 1 \implies$

$$\begin{aligned} \dot{V} &\leq -\zeta\,\|x\|^2 + \frac{\sqrt{29}\delta}{8}\|x\| \\ &\leq -(1-\theta)\zeta\|x\|^2 \quad \forall\|x\| \geq \mu = \frac{\sqrt{29}\delta}{8\zeta\theta} \quad 0 < \theta < 1 \end{aligned}$$

- We saw earlier that $|x_2|^2$ is bounded on Ω_c by $\frac{96c}{29} \implies$ if $\beta \leq .4(1-\zeta)/c$ and δ is so small $\implies B_\mu \subset \Omega_c$ and all trajectories starting inside Ω_c remain for all future time in Ω_c.
- The conditions of the theorem is satisfied in $\Omega_c \implies$ Sol. of the perturbed system is u.u.b. with bound

$$b = \frac{\sqrt{29}\delta}{8\zeta\theta}\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$$

Nonvanishing Perturbation

- The following lemma provides conditions for u.a.s rather than e.s of origin
- **Lemma:** *Let $x = 0$ be u.a.s Equ. pt of the* **nominal system** *($\dot{x} = f(t, x)$). Let $V(t, x)$ be a Lap fcn. of he* **nominal system***that satisfies:*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|), \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(\|x\|), \|\frac{\partial V}{\partial x}\| \leq \alpha_4(\|x\|)$$

in $[0, \infty) \times D$, where $D = \{x \in \mathcal{R}^n \| \|x\| < r\}$, and α_i, $i = 1, ..., 4$ are class $\mathcal{K}$ fcns. Suppose the perturbation term satisfies

$$\|g(t,x)\| \leq \delta < \frac{\theta\alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{\alpha_4(r)} \quad \forall t \geq 0, \ \forall x \in D, \ 0 < \theta < 1$$

then $\forall \|x(t_0)\| < \alpha_2^{-1}(\alpha_1(r))$, the sol. of the perturbed system satisfies

$$\begin{aligned} \|x(t)\| &\leq \beta(\|x(t_0)\|, t - t_0) t_0 \leq t < t_0 + T \\ \& \quad \|x(t)\| &\leq p(\delta) \ \forall \ t \geq t_0 + T \end{aligned}$$

for some finite time T , and class $\mathcal{KL}$ fcn β, where p is class $\mathcal{K}$ fcn of δ: $p(\delta) = \alpha^{-1}(\alpha_2(\alpha_3^{-1}(\frac{\delta\alpha_4(r)}{\theta})))$

Nonvanishing Perturbation

- Note that there is no counterpart for more general case than u.a.s.
- In the case of e.s., upper bound of perturbation term: $\delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r$
  - $\delta \to \infty$ as $r \to \infty$
  - $\therefore$ for all unifobounded disturbance, the solution of perturbed system is uniformly bounded
- In the case of a.s., upper bound of perturbation term: $\delta < \frac{\theta\alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{\alpha_4(r)}$
  - No prediction on limit of δ as $r \to \infty$ without further info about class $\mathcal{K}$ fcns.