Signals and Systems
Lecture 5: Fourier Transform

Farzaneh Abdollahi

Department of Electrical Engineering

Amirkabir University of Technology

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CT Fourier Transform
Convergence of CT FT
CT FT Properties

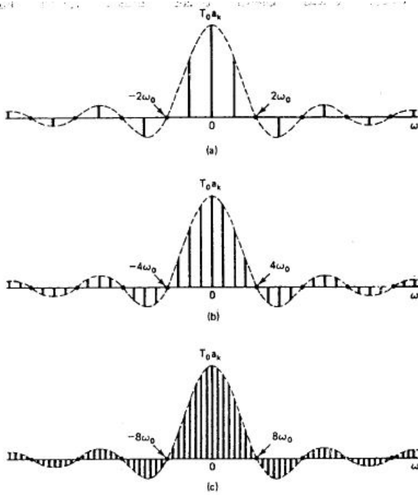
DT Fourier Transform
Convergence of DT FT
DT Fourier Transform for Periodic Signals
DT FT Properties

CT Fourier Transform

- Fourier series was defined for periodic signals
- Aperiodic signals can be considered as a periodic signal with fundamental period ∞!
- $T_0 \longrightarrow \infty \rightsquigarrow \omega_0 \longrightarrow 0$
  - The harmonics get closer
  - summation $(\sum)$ is substituted by $(\int)$
  - Fourier series will be replaced by Fourier transform

Example:

- Consider a periodic square wave:
  $$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T_0/2 \end{cases}$$
- The fourier series coefficient:
  $$a_k = \frac{2sin(k\omega_0 T_1)}{k\omega_0 T_0} \rightsquigarrow$$
  $$T_0 a_k = \frac{2sin(\omega T_1)}{\omega}\Big|_{\omega = k\omega_0}$$
- $T_0 \uparrow (\omega_0 \downarrow) \rightsquigarrow$ number of samples $\uparrow$
- $T_0 \to \infty \rightsquigarrow$ the Fourier series coefficients approaches the envelop



Fourier coefficients and their envelope for the periodic squa wave: (a) $T_0 = 4T_1$; (b) $T_0 = 8T_1$; (c) $T_0 = 16T_1$.

Fourier Transform (FT)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Convergence of CT FT
  - $x(t)$ should be square integrable $\int_{-\infty}^{\infty}|x(t)|^2dt < \infty$
    - Let us define the Fourier representation of $x(t)$ by $\hat{x}(t) = \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$
    - $\therefore$ the above condition can guarantee the energy of the error $(e(t) = x(t) - \hat{x}(t))$ is zero, except some individual values of t
    - i.e. $\int_{-\infty}^{\infty}|e^2(t)|dt < \infty$

Convergence of CT FT

- To ensure $x(t) = \hat{x}(t)$ for any t (except discontinuities which will be the average value of discontinuity) the following Dirichlet conditions should be satisfied:
  1. Absolute integrality of $x(t)$: $\int_{-\infty}^{\infty} |x(t)|dt < \infty$
  2. Within any finite interval $x(t)$ should have finite max and min points
  3. Within any finite interval $x(t)$ should have finite discontinuities; the discontinuities should be finite.
- Exercise: Does Gibbs phenomena applicable for FT?
- **Note that:** Dirichlet conditions are sufficient conditions for FT convergence
- If impulse function is permitted in transform
  - Periodic signals which are neither absolute integrable nor square integrable over infinite interval has FT
  - Of course in a finite interval (a period) they should be integrable and square integrable)
- Hence FT and Fs can be considered in a common framework;)

- For impulse fcn:
  - $X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$
  - $\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega$
- $x(t) = \delta(t - t_0) \leadsto X(j\omega) = e^{-j\omega t_0}$
- FT for Periodic Signals
  - $X(j\omega) = 2\pi\delta(\omega - \omega_0) \leadsto x(t) = e^{j\omega_0 t}$
  - Now for $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \leadsto x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ (which is Fourier series representation of a periodic signal)
  - Therefore
    - By having a_k, $X(j\omega)$ is obtained: $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$
    - By having $X(j\omega)$, a_k is obtained: $a_k = \frac{1}{T} X(j\omega)|_{\omega = k\omega_0}$

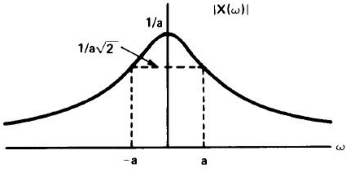
Some CT FT Properties

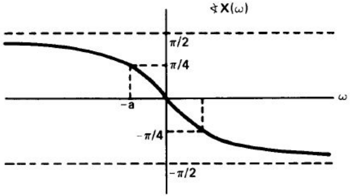
- Linearity: $ax(t) + by(t) \Leftrightarrow aX(j\omega) + bY(j\omega)$
- Time Shifting: $x(t - t_0) \Leftrightarrow e^{-j\omega t_0} X(j\omega)$
  - No change in amplitude: $|e^{-j\omega t_0} X(j\omega)| = |X(j\omega)|$
  - Linear change in phase: $\measuredangle e^{-j\omega t_0} X(j\omega) = \measuredangle X(j\omega) - \omega t_0$
- Integration and Differentiation:
  - $\frac{dx(t)}{dt} \Leftrightarrow j\omega X(j\omega)$
  - $\int_{-\infty}^{t} x(\tau) d\tau \Leftrightarrow \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(j\omega)$
- Time/Frequency Scaling: $x(at) \Leftrightarrow \frac{1}{|a|} X(\frac{j\omega}{a})$
  - $\mathcal{F}\{e^{-at}u(t)\} = \frac{1}{a+j\omega} = \frac{1}{a} \frac{1}{1+j(\frac{\omega}{a})}$

- Conjugate and Conjugate Symmetry: $x^*(t) \Leftrightarrow X^*(j\omega)$
  - Real $x(t) \Leftrightarrow X(-j\omega) = X^*(j\omega)$
    - Polar representation $X(j\omega) = |X(j\omega)|e^{j\measuredangle X(j\omega)}$
      - $|X(-j\omega)| = |X(j\omega)|$ (even fcn)
      - $\measuredangle X(-j\omega) = -\measuredangle X(j\omega)$ (odd fcn)
    - Rectangular representation $X(j\omega) = \mathcal{R}e\{X(j\omega)\} + j\mathcal{I}m\{X(j\omega)\}$
      - $\mathcal{R}e\{X(-j\omega)\} = \mathcal{R}e\{X(j\omega)\}$ (even fcn)
      - $\mathcal{I}m\{X(-j\omega)\} = -\mathcal{I}m\{X(j\omega)\}$ (odd fcn)
    - Real and even $x(t) = x(-t) \Leftrightarrow X(j\omega) = X(-j\omega) = X^*(j\omega)$ (real and even $X(j\omega)$)
    - Real and odd $x(t) = -x(-t) \Leftrightarrow X(-j\omega) = -X(j\omega) = -X^*(j\omega)$ (purely imaginary and odd $X(j\omega)$),
    - Even part of $x(t) \Leftrightarrow \mathcal{R}e\{X(j\omega)\}$ (show it!)
    - Odd part of $x(t) \Leftrightarrow j\mathcal{I}m\{X(j\omega)\}$ (show it!)

Example

- $\mathcal{F}\{e^{-at}u(t)\} = \frac{1}{a+j\omega}, \quad a > 0$



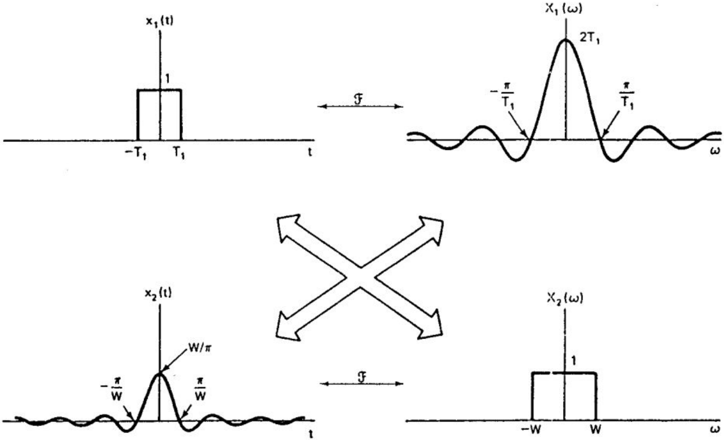


- Duality:
  - Reconsider TF:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

  - They are similar But not quite identical!
  - We can find a duality relation between them
- **Example:**
  - $\frac{dx(t)}{dt} \Leftrightarrow j\omega X(j\omega)$
  - $-jtx(t) \Leftrightarrow \frac{dX(j\omega)}{d\omega}$

Duality: Example



Some CT FT Properties

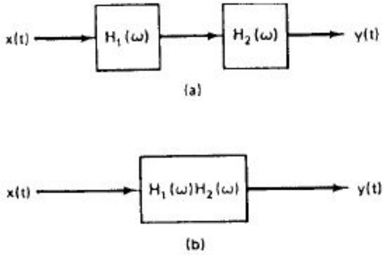
- Parseval's Relation: $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$
  - Total energy in obtained by
    - computing energy per unit time then integrating over all time OR
    - computing energy per unit frequency and integrating over all frequencies
  - $|X(j\omega)|^2$ is called energy-density spectrum.
- Convolution: $y(t) = x(t) * h(t) \Leftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$
  - This property can be used for filtering input signal in frequency domain.

FT for LTI Systems

- FT of impulse response, $H(j\omega)$,is called Frequency Response of the system
- Plays a key role in LTI system analyzing
- Convergence Condition of FT in LTI Systems
  - The LTI system should be stable
  - i.e., impulse response should be absolute integrable: $\int_{-\infty}^{\infty} |h(t)|dt < \infty$
  - Note that this is one of Drichlet's conditions.
  - Assume other two Drichlet's conditions are satisfied (This happens for all practical systems)
  - $\therefore$ Only stable LTI systems can be analyzed by FT
  - Question: How can we analyze unstable LTI systems?

FT for LTI Systems

- The overall Freq. Res. (Frequency Response) of two cascade system is product of individual Freq. responses:



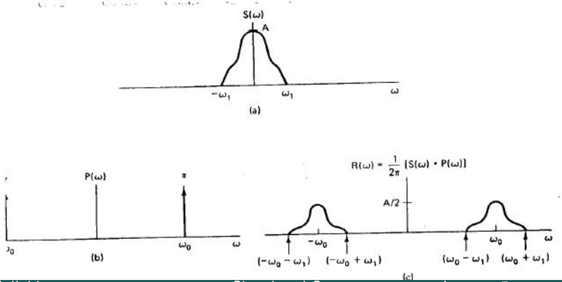
- **Example:** Consider $h(t) = \delta(t - t_0) \Leftrightarrow H(j\omega) = e^{-j\omega t_0}$
  - $Y(j\omega) = H(j\omega)X(j\omega) = e^{-j\omega t_0}X(j\omega) \Leftrightarrow y(t) = x(t - t_0)$

FT for LTI Systems

- **Example:** $y(t) = \frac{dx(t)}{dt} \Leftrightarrow Y(j\omega) = j\omega X(j\omega)$
  - $H(j\omega) = j\omega$
  - Differentiating system increases the magnitude by ω, and increase the phase by $\pi/2, (j = e^{j\frac{\pi}{2}})$
  - $\frac{d}{dt} sin\omega_0 t = \omega_0 cos(\omega_0 t) = \omega_0 sin(\omega_0 t + \frac{\pi}{2})$
- **Example:** Consider $x(t) = e^{-at}u(t), \; h(t) = e^{-bt}u(t)$
  - $X(j\omega) = \frac{1}{a+j\omega}, H(j\omega) = \frac{1}{b+j\omega}$
  - $Y(j\omega) = \frac{1}{(a+j\omega)(b+j\omega)} = \frac{1}{(b-a)(a+j\omega)} + \frac{1}{(a-b)(b+j\omega)}$
  - $y(t) = \frac{1}{b-a}\{e^{-at}u(t) - e^{-bt}u(t)\}$

FT Properties

- Multiplication: $r(t) = s(t)p(t) \Leftrightarrow R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$
  - Multiplication of two signals is using one signal to scale (modulate) the amplitude of another one.
  - $\therefore$ Multiplication of two signals is called amplitude modulation.
  - **Example:** $p(t) = cos(\omega_0)t \Leftrightarrow P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
    - $r(t) = s(t)p(t) \Leftrightarrow R(j\omega) = \frac{1}{2}S(j(\omega - \omega_0)) + \frac{1}{2}S(j(\omega + \omega_0))$
    - signal of $s(t)$ is preserved by multiplying it by sinusoidal signal, its information is only shifted to the higher frequency.(sinusoidal amplitude modulation).



DT Fourier Transform

- Similar to CT, aperiodic signals for DT can be considered as a periodic signal with fundamental period ($N \to \infty$):
  - Consider $x[n]$ is aperiodic and has values for $-N_1 \leq n \leq N_2$
  - Define a periodic signal $\tilde{x}[n]$ with fundamental period N which is identical to $x[n]$ in $-N_1 : N_2$ interval
  - as $N \to \infty \quad x[n] = \tilde{x}[n]$
  - $\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}, \quad a_k = \frac{1}{N}\sum_{n=<N>} \tilde{x}[n]e^{-jk(2\pi/N)n}$
  - $a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} \tilde{x}[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk(2\pi/N)n}$
  - Now define $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \rightsquigarrow a_k = \frac{1}{N}X(e^{jk\omega_0})$
  - also $\frac{1}{N} = \frac{\omega_0}{2\pi}$
  - $\therefore \tilde{x}[n] = \frac{1}{2\pi}\sum_{k=<N>} X(e^{j\omega_0})e^{j\omega_0 n}\omega_0$
  - $N\uparrow \rightsquigarrow \omega_0\downarrow$
  - When $N \to \infty \rightsquigarrow$ summation ($\sum$) is substituted by ($\int$)
  - Moreover, $X(e^{j\omega})$ and $e^{j\omega n}$ are periodic with period 2π
  - $\therefore N \to \infty \rightsquigarrow x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$

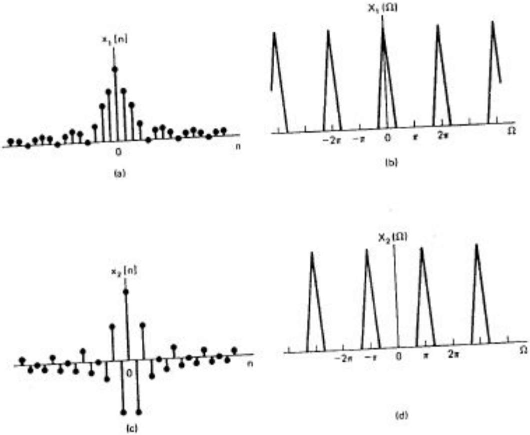
DT Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- The main differences between CT and DT fourier transforms:
  1. In DT, $X(e^{j\omega})$ is periodic
  2. In DT, the integral of the synthesis equation is finite.
- These properties are similar to DT Fourier Series and they are due to the fact that DT complex exponentials are periodic with 2π

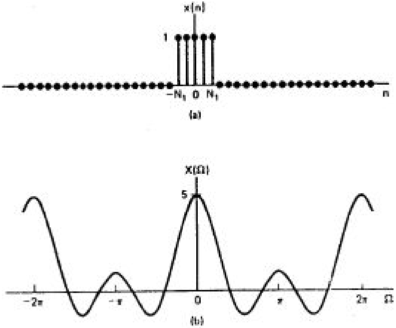
- Reminder: For $e^{j\omega n}$, $\omega = 0$ and $\omega = 2\pi$ leads to the same signal
- Signals at frequencies near even multiple of π are slowly varying
- Signals at frequencies near odd multiple of π are fast varying



Example

- Consider a rectangular pulse: $x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases}$
- For $N_1 = 2:\quad X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$
  - Using Euler rule and doing some manipulations yield to $X(e^{j\omega}) = \frac{sin\omega(N_1+\frac{1}{2})}{sin(\omega/2)}$ (Show it!)
  - This is DT counterpart of sinc function that is obtained from FT of CT rectangular pulse signal
  - The main difference is that the sinc function in DT is periodic but in CT it is aperiodic

Example Cont'd



Convergence of DT FT

- To derive DT FT we considered a $x[n]$ with finite duration.
- BUT DT FT is valid for signals with infinite duration as well (such as unit step and etc.)
- The conditions on $x[n]$ to guarantee convergence of $X(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ is similar to CT FT:
  - $x[n]$ has finite energy: $\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$
  - OR $x[n]$ is absolute summable: $\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$

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  - OR $x[n]$ is absolute summable: $\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$
- However No convergence condition is required regarding the synthesis equation $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$
  - Since the integral is over a finite interval
  - This property is similar to DT Fourier series

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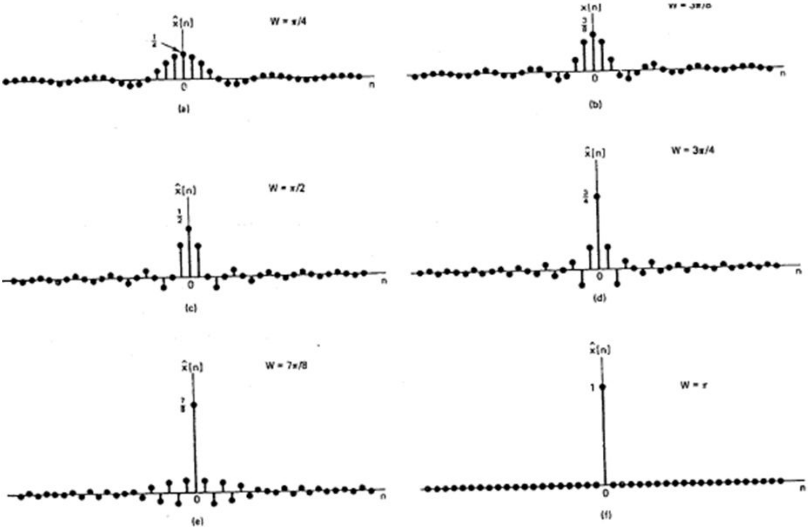
  - Since the integral is over a finite interval
  - This property is similar to DT Fourier series
- We would expect No Gibbs phenomenon behavior for DT FT

Example

- Consider $x[n] = \delta[n]$
- $X(e^{j\omega}) = 1$
- FT of impulse response provides equal contribution at all frequencies
- We can also define $\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n} d\omega = \frac{\sin Wn}{\pi n}$
- and obtain $x[n]$ by increasing W to π
- We can see
  - similar to CT, $W \uparrow \rightsquigarrow$ oscillation $\uparrow$
  - despite of CT, $W \uparrow \rightsquigarrow$ the amplitude of $\hat{x}[0] \uparrow$ and amplitude of oscillations $\downarrow$
- $\therefore$ There is no Gibbs phenomenon

Example Cont'd



DT FT For Periodic signals

- Similar to CT, DT FT of $x[n] = e^{j\omega_0 n}$ is a signal of impulse function
- Since DT FT is periodic, DT FT of $x[n]$ should have impulses at $\omega_0, \omega_0 \pm 2\pi, \omega_0 \pm 4\pi$ and so on:

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

- $\therefore$ DT FT of $x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$ is:

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} \sum_{k=<N>} 2\pi a_k \delta(\omega - \frac{2k\pi}{N} - 2\pi l)$$

- a_k is periodic, $a_0 = a_N, a_1 = a_{N+1}, \ldots \rightsquigarrow$ the equation can be simplified to

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})$$

- $\therefore$ FT of a periodic Signal can be obtained from its FS coefficients :)

Example

- Consider $x[n] = \sin \omega_0 n = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$ with $\omega_0 = \frac{2\pi}{7}$
- $\therefore X(e^{j\omega_0}) = \sum_{l=-\infty}^{+\infty} \frac{\pi}{j} \delta(\omega - \frac{2\pi}{7} - 2\pi l) - \sum_{l=-\infty}^{+\infty} \frac{\pi}{j} \delta(\omega + \frac{2\pi}{7} - 2\pi l)$

Some DT FT Properties

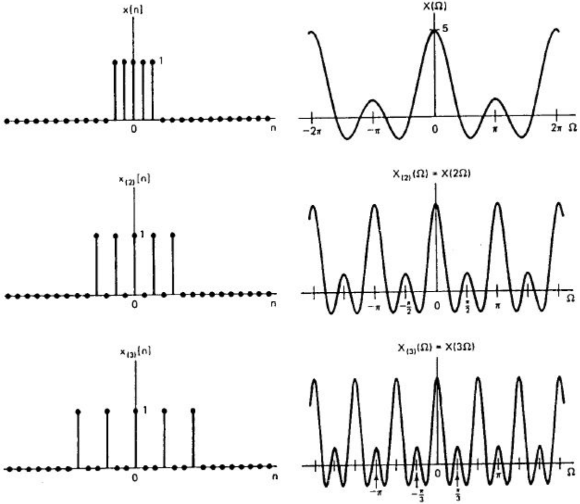
- Periodicity: DT FT is always periodic in ω with period 2π: $X(e^{j\omega_0}) = X(e^{j\omega_0} + 2\pi)$
- Linearity: $ax[n] + by[n] \Leftrightarrow aX(e^{j\omega}) + bY(e^{j\omega})$
- Time/Frequency Shifting:
  $x[n - n_0] \Leftrightarrow e^{-j\omega n_0} X(e^{j\omega})$
  $e^{-j\omega_0 n} x[n] \Leftrightarrow X(e^{j(\omega - \omega_0)})$
- Differencing and Summation:
  - $x[n] - x[n-1] \Leftrightarrow (1 - e^{-j\omega})X(e^{j\omega})$
  - $\sum_{m=-\infty}^{n} x[m] \Leftrightarrow \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
- Time Reversal: $x[-n] \Leftrightarrow X(e^{-j\omega})$

Some DT FT Properties

- Time Expansion:
  - For $x[an]$, a should be integer
  - Therefore, $a < 1$ does not necessarily make the signal slow down
  - To speed up the original signal we cannot use $a > 1$ since it does not keep all the original signal elements
  - For instance for $a = 2$, $x[2n]$ just keeps the even samples of $x[n]$
  - So let us define signal
    $$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if n=r k, where r,k are integer} \\ 0 & \text{otherwise} \end{cases}$$
  - Therefore by placing $k-1$ zeros between successive samples of original signal, a kind of slowing down signal is defined
  - $X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}$
  - $x_{(k)}[rk] = x[r] \rightsquigarrow X_{(k)}(e^{j\omega}) = X(e^{jk\omega})$
  - $\therefore x_{(k)}[n] \Leftrightarrow X(e^{jk\omega})$

Example

- For a rectangular pulse signal:



- Conjugate and Conjugate Symmetry: $x^*[n] \Leftrightarrow X^*(e^{-j\omega})$
  - Real $x[n] \Leftrightarrow X(e^{-j\omega}) = X^*(e^{j\omega})$
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    - Rectangular representation $X(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\} + j\mathcal{I}m\{X(e^{j\omega})\}$
      - $\mathcal{R}e\{X(e^{-j\omega})\} = \mathcal{R}e\{X(e^{j\omega})\}$ (even fcn)
      - $\mathcal{I}m\{X(e^{-j\omega})\} = -\mathcal{I}m\{X(e^{j\omega})\}$ (odd fcn)
    - Real and even $x[n] = x[-n] \Leftrightarrow X(e^{j\omega}) = X(e^{-j\omega}) = X^*(e^{j\omega})$ (real and even $X(e^{j\omega})$)
    - Real and odd $x[n] = -x[-n] \Leftrightarrow X(e^{-j\omega}) = -X(e^{j\omega}) = -X^*(e^{j\omega})$ (purely imaginary and odd $X(e^{j\omega})$),
    - even part of $x[n] \Leftrightarrow \mathcal{R}e\{X(e^{j\omega})\}$ (show it!)
    - Odd part of $x[n] \Leftrightarrow j\mathcal{I}m\{X(e^{j\omega})\}$ (show it!)

Some DT FT Properties

- Differentiation in Frequency: $nx[n] \Leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$
- Parseval's Relation: $\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$
- Convolution: $y[n] = x[n] * h[n] \Leftrightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$
  - This property can be used for filtering input signal in frequency domain.
- Multiplicity: $y[n] = x_1[n]x_2[n] \Leftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$
  - The left side equation is called periodic convolution

Dualities

	CT		DT	
	Time Domain	Freq. Domain	Time Domain	Freq. Domain
FS	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ Continuous time Periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\, dt$ Discrete freq. Aperiodic in freq.	$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$ Discrete time Periodic in time	$a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk(\frac{2\pi}{N})t}$ Discrete freq. Periodic in freq.
FT	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$ Continuous time Aperiodic in time	$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$ Continuous freq. Aperiodic in freq.	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$ Discrete time Aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$ Continuous freq. Periodic in Freq.

FT for DT LTI Systems

- A DT LTI system can be expressed as: $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$
- Now take DT FT: $\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$
- Frequency response $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$