Nonlinear Control
Lecture 6: Stability Analysis III

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Input-to-State Stability

- Consider the system $$\dot{x} = f(t, x, u) \qquad (1)$$

  where $f : [0, \infty) \times R^m \longrightarrow R^n$ is piecewise continuous, local lip in x and u. $u(t)$ is p.c. and bounded fcn $\forall t \geq 0$.
- Suppose the Equ. pt. of the unforced system below is **g.u.a.s.** $$\dot{x} = f(t, x, 0) \qquad (2)$$
- What can be said about the behavior of the forced system in the presence of a bounded input $u(t)$.
- For an LTI system: $$\dot{x} = Ax + Bu$$

  where A is Hurwitz, the solution satisfies:
  $$\|x(t)\| \leq ke^{-\lambda(t-t_0)} \|x(t_0)\| + \frac{k\|B\|}{\lambda} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\|$$
  - Zero-input response decays to zero
  - Zero-state response remains **bounded** for **bounded input**

Input-to-State Stability

- Can this conclusion be extended to nonlinear system (1)?
  - The answer in general is no, for instance:
    $$\dot{x} = -3x + (1 + 2x^2)u$$
    when $u = 0$, the origin is **g.e.s**
  - However, with $x(0) = 2$ and $u(t) \equiv 1$, $x(t) = (3 - e^t)/(3 - 2e^t)$ is unbounded and have a finite scape time.
- View the system $\dot{x} = f(t, x, u)$ as a perturbation of the unforced system $\dot{x} = f(t, x, 0)$.
- Suppose there exists a Lyap. fcn for the unforced system and calculate $\dot{V}$ in the presence of u
- Since u is bounded, it may be possible to show that $\dot{V}$ is n.d. outside of a ball with radius μ where μ depends on sup $\|u\|$.
- This is possible, for instance if the function $f(t, x, u)$ is Lip. in u, i.e.
  $$\|f(t, x, u) - f(t, x, 0)\| \leq L\|u\|$$

Input-to-State Stability

- Having shown $\dot{V}$ is negative outside of a ball, ultimate boundedness theorem can be used, i.e.
  - $\|x(t)\|$ is bounded by a class $\mathcal{KL}$ fcn $\beta(\|x(t_0)\|, t-t_0)$ over $[t_0, t_0+T]$ and by a class $\mathcal{K}$ fcn $\alpha_1^{-1}(\alpha_2(\mu))$ for $t \geq t_0 + T$
- Hence, $\|x(t)\| \;\leq\; \beta(\|x(t_0)\|, t-t_0) + \alpha^{-1}(\alpha_2(\mu)) \;\; \forall\, t \geq \; t_0$
- **Definition:** The system (1) is said to be **input-to-state stable** if there exist a class $\mathcal{KL}$ fcn β and a class $\mathcal{K}$ fcn γ s.t. for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \;\geq\; t_0$ and satisfies:

$$\|x(t)\| \;\leq\; \beta(\|x(t_0)\|, t-t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)$$

  - If $u(t)$ converges to zero as $t \;\longrightarrow\; \infty$, so does $x(t)$.
  - with $u(t) \equiv 0$, the above equation reduces to:

$$\|x(t)\| \;\leq\; \beta(\|x(t_0)\|, t-t_0)$$

implying the origin of unforced system is **g.u.a.s.**

Input-to-State Stability

- **Sufficient** condition for input-to-state stability:
- **Theorem:** *Let* $V : [0,\infty) \times R^n \longrightarrow R$ *be a cont. diff. fcn. s.t.*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) \leq -W_3(x), \quad \forall \ \|x\| \geq \rho(\|u\|) > 0$$

$\forall (t,x,u) \in [0,\infty) \times R^n \times R^m$ where α_1 and α_2 are class $\mathcal{K}_\infty$ fcns, ρ is a class $\mathcal{K}$ fcn, and $W_3(x)$ is a cont. p.d. fcn. on R^n. Then, the system (1) is input-to-state stable with

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$$

Input-to-State Stability

- **Lemma:** *Suppose $f(t,x,u)$ is cont. diff. and* **globally** *Lip. in (x,u), uniformly in t. If the unforced system has a* **globally exponentially stable** *Equ. pt. at the origin, then the system (1) is* **input-to-state stable** *(ISS).*
- **Proof:**
  - View the forced system as a perturbation to unforced system
  - The converse theorem implies that the unforced system has a Lyap. fcn satisfying the **g.e.s** conditions.
  - The perturbation terms satisfies the Lip. cond. $\forall\ t \geq 0$ and $\forall\ (x,u)$.
  - Hence, $\dot{V}$ along the trajectories of forced system (1):

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) + \frac{\partial V}{\partial x}[f(t,x,u) - f(t,x,0)] \\
&\leq -c_3\|x\|^2 + c_4\|x\|L\|u\| = -c_3(1-\theta)\|x\|^2 - c_3\theta\|x\|^2 \\
&+ c_4\|x\|L\|u\|, \quad 0 < \theta < 1
\end{aligned}$$

Input-to-State Stability

$$\therefore \dot{V} \;\leq\; -c_3(1-\theta)\|x\|^2 \forall\ \|x\| \;\geq \frac{c_4 L\|u\|}{c_3\theta},\ \ \forall\ (t,x,u)$$

- The conditions of previous theorem are satisfied with:

$$\alpha_1(r) = c_1 r^2,\ \alpha_2(r) = c_2 r^2,\ \rho(r) = (c_4 L/c_3\theta) r$$

- $\therefore$ The system is input-to-state stable with $\gamma(r) = \sqrt{c_2/c_1}(c_4 L/c_3\theta) r$
- The previous lemma relies on **globally Lip**. fcn f and global **exponential stability** of the origin of the unforced system for **ISS**.

Input-to-State Stability

- **Example 1:** $$\dot{x} = -\frac{x}{1+x^2} + u = f(x,u)$$

  has a globally Lip. f since $\frac{\partial f}{\partial x} = -\frac{1-x^2}{(1+x^2)^2}$ and $\frac{\partial f}{\partial u} = 1$ and are globally bounded.
- The origin of unforced system $\dot{x} = -\frac{x}{1+x^2}$ is $g.a.s(V = x^2/2 \implies \dot{V} = -\frac{x^2}{1+x^2}$ **n.d.** for all x)
- The system is locally **e.s** because of the linearized system $\dot{x} = -x$
- However, the system is not **g.e.s**

$$u \equiv 1, f(x,u) \geq 1/2 \implies x(t) \geq x(t_0) + t/2 \ \forall\, t \geq t_0$$

- $\therefore$ This is not ISS
- If **g.e.s.** and globally Lip. conds. are not satisfied, then we can use previous theorem to show **ISS** (i.e. find a region $\|x\| \geq \mu$ in which $\dot{V} < 0$)

Input-to-State Stability

- **Example 2:** $$\dot{x} = -x^3 + u$$

  has a **g.a.s.** Equ. pt. at the origin when $u = 0$.
- Let $V = x^2/2$, then $\dot{V}$ can be written as:
  $\dot{V} = -x^4 + ux \;=\; -(1-\theta)x^4 - \theta x^4 + xu$
  $\leq \; -(1-\theta)x^4, \; \forall |x| \; \geq \; \left(\frac{|u|}{\theta}\right)^{1/3} \quad 0 \; < \; \theta \; < \; 1.$
- The system is ISS with $\gamma(r) = (r/\theta)^{1/3}$
- **Example 3:** $$\dot{x} = f(x,u) = -x - 2x^3 + (1+x^2)u^2$$

  has a **g.e.s.** Equ. pt. at the origin when $u = 0$.
- However, f is not globally Lip. Let $V = x^2/2$, then:
  $$\dot{V} = -x^2 - 2x^4 + x(1+x^2)u^2 \;\leq\; -x^4, \; \forall\, |x| \;\geq\; u^2$$
- The system is ISS with $\gamma(r) = r^2$

Stability of Cascade System

- Consider the cascade system

$$\dot{x}_1 = f_1(t, x_1, x_2), \quad f_1 : [0, \infty) \times R^{n_1} \times R^{n_2} \longrightarrow R^{n_1} \tag{3}$$

$$\dot{x}_2 = f_2(t, x_2), \quad f_2 : [0, \infty) \times R^{n_2} \longrightarrow R^{n_2} \tag{4}$$

where f_1 and f_2 are p.c. in t and locally Lip. in $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

- Suppose $\dot{x}_1 = f_1(t, x_1, 0)$ and (4) both have **g.u.a.s.** Equ. pt. at $x_1 = 0$ and $x_2 = 0$.
- Under what condition the origin of the cascade system is also **g.u.a.s.**?
- The condition is that (3) should be **ISS** with x_2 viewed as input.
- **Lemma:** *Under the assumption given above, if the system (3) with x_2 as input, is* **ISS** *and the origin of (4) is* **g.u.a.s.**, *then the origin of the cascade system (3) and (4) is* **g.u.a.s.**

Input-Output Stability

- The foundation of input-output (I/O) approaches to nonlinear systems can be found in 1960's by Sandberg and Zames
- An input-output model relates output to input with no knowledge of the internal structure (state equation).

$$y = Hu, \quad u : [0, \infty) \to \mathcal{R}^m$$

- The norm function $\|u\|$ should satisfy the three properties
  1. $\|u\| = 0$ iff $u = 0$ and it is strictly positive otherwise
  2. scaling property $\forall a > 0, u \Rightarrow \|au\| = a\|u\|$
  3. triangular inequality: $\forall u_1, u_2, \ \|u_1 + u_2\| \le \|u_1\| + \|u_2\|$
- Example:

$$\|u\|_{\mathcal{L}_\infty^m} = \sup_{t\ge 0} \|u\| < \infty$$

$$\|u\|_{\mathcal{L}_2^m} = \sqrt{\int_0^\infty u^T(t)u(t)dt} < \infty$$

$$\|u\|_{\mathcal{L}_p^m} = \int_0^\infty \|u\|^p dt)^{1/p} < \infty, \quad 1 \le p < \infty$$

Input-Output Stability

- Stable system: any "well-behaved" input generate a "well-behaved" output
- Extended space: $\mathcal{L}_e^m = \{u | u_\tau \in \mathcal{L}^m, \forall \tau \in [0, \infty)\}$
  - where u_τ is a truncation of u: $u_\tau(t) = \begin{cases} u(t) & 0 \leq t \leq \tau \\ 0 & t > \tau \end{cases}$
  - It allows us to deal with unbounded "ever-growing" signals
  - **Example:** $u(t) = t \notin \mathcal{L}_\infty$ but $u_\tau(t) \in \mathcal{L}_{\infty e}$
- Casuality: mapping $H : \mathcal{L}_e^m \rightarrow \mathcal{L}_e^q$ is causal if the output $(Hu)(t)$ at any time t depends only on the value of the input **up to time** t

$$(Hu)_\tau = (Hu_\tau)_\tau$$

Input-Output Stability

- **Definition**: *A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is $\mathcal{L}$* **stable** *if there exist a class $\mathcal{K}$ function α, defined on $[0,\infty)$ and a nonneg const. β s.t.*

$$\|(Hu)_\tau\|_\mathcal{L} \leq \alpha(\|u_\tau\|_\mathcal{L}) + \beta, \quad \forall u \in \mathcal{L}_e^m, \tau \in [0,\infty) \tag{5}$$

It is **finite-gain** *$\mathcal{L}$ stable if there exist nonneg. const. γ and β s.t.*

$$\|(Hu)_\tau\|_\mathcal{L} \leq \gamma\|u_\tau\|_\mathcal{L} + \beta, \quad \forall u \in \mathcal{L}_e^m, \tau \in [0,\infty) \tag{6}$$

  - β is bias term $\rightsquigarrow$ allows Hu does not vanish at $u = 0$
  - In finite-gain $\mathcal{L}$ stability , the smallest possible γ is desired to satisfy (6)
  - $\mathcal{L}_\infty$ stability is bounded-input-bounded-output stability.

- **Example 4:** $y(t) = h(u) = a + b\tanh cu = a + b\frac{e^{cu}-e^{-cu}}{e^{cu}+e^{-cu}}$, for $a, b, c \geq 0$
  - using the fact: $\acute{h}(u) = \frac{4bc}{(e^{cu}+e^{-cu})^2} \leq bc, \quad \forall\ u \in R$
  - $\therefore\ |h(u)| \leq a + bc|u|, \ \forall\ u \in R$
  - it is finite gain $\mathcal{L}_\infty$ stable with $\gamma = bc, \quad \beta = a$

Input-Output Stability

- **Example:** $y(t) = h(u) = u^2$
  - $\sup_{t\geq 0} |h(u(t))| \leq (\sup_{t\geq 0} |u(t)|)^2$
  - $\therefore$ it is $\mathcal{L}_\infty$ stable with $\beta = 0,\ \alpha(r) = r^2$
  - But it is **not** finite-gain $\mathcal{L}_\infty$ stable since $h(u)$ can not be bounded by a straight line of the form $|h(u)| \leq \gamma|u| + \beta$ for all $u \in R$
- **Example:** $y = \tan u$
  - $y(t)$ is defined only for $|u(t)| < \frac{\pi}{2},\ \forall t \geq 0 \rightsquigarrow$ it is not $\mathcal{L}_\infty$ stable
  - If we restrict $|u| \leq r \leq \frac{\pi}{2} \rightsquigarrow |y| \leq (\frac{\tan r}{r})|u|$
  - it is small-gain $\mathcal{L}$ stable
- Definition: *mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is* **small-signal $\mathcal{L}$ stable**/**small-signal finite-gain $\mathcal{L}$ stable** *if there exist r s.t. inequality (5)/(6) is satisfied for all $u \in \mathcal{L}_e^m$ with $\sup_{0\leq t\leq \tau} \|u\| \leq r$*

$\mathcal{L}$ Stability of State Models

- What can we say about I/O stability based on the formalism of Lyapunov stability?
- Consider

$$\begin{aligned} \dot{x} &= f(t, x, u) \\ y &= h(t, x, u) \end{aligned} \tag{7}$$

  - $x \in R^m, \;\; y \in R^q$
  - $f : [0, \infty) \times D \times D_u \longrightarrow R^n$ is p.c. in t, locally Lipshitz in (x, u)
  - $h : [0, \infty) \times D \times D_u \longrightarrow R^q$ p.c. in t and cont. in (x, u)
  - $D \subset R^n$ is a domain containing $x = 0$
  - $D_u \subset R^m$ is a domain containing $u = 0$
  - Assume the unforced system $\dot{x} = f(t, x, 0)$ is u.a.s (or e.s)

- Theorem: *Consider the system (7) and take $r_u, r > 0$ s.t. $\{\|x\| \leq r\} \subset D$ and $\{\|u\| \leq r_u\} \subset D_u$. Suppose that*
  - *$x = 0$ is an e.s. Equ. point of $\dot{x} = f(t, x, 0)$ and there is a Lyap. fcn $V(t, x)$ and positive const c_i, $i = 1, \ldots, 4$ that*

$$c_1 \|x\|^2 \leq V(t,x) \leq c_2 \|x\|^2, \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) \leq -c_3 \|x\|^2$$

$$\|\frac{\partial V}{\partial x}\| \leq c_4 \|x\| \qquad \forall (t,x) \in [0,\infty) \times D,$$

  - *$\forall (t, x, u) \in [0, \infty) \times D \times D_u$ and for some nonneg. const. L, η_1, and η_2:*

$$\|f(t,x,u) - f(t,x,0)\| \leq L\|u\|, \quad \|h(t,x,u)\| \leq \eta_1 \|x\| + \eta_2 \|u\|$$

- *Then for each $\|x_0\| \leq r\sqrt{c_1/c_2}$ the system is small-signal finite-gain $\mathcal{L}_p$ stable for each $p \in [1, \infty]$. In particular, for each $u \in \mathcal{L}_{pe}$ with $\sup_{0 \leq t \leq \tau} \|u\| \leq \min\{r_u, c_1 c_3 r/(c_2 c_4 L)\}$ the output satisfies:*

$$\|y_\tau\|_{\mathcal{L}_p} \leq \gamma \|u_\tau\|_{\mathcal{L}_p} + \beta, \quad \tau \in [0, \infty)$$

$$\gamma = \eta_2 + \frac{\eta_1 c_2 c_4 L}{c_1 c_3}, \ \beta = \eta_1 \|x_0\| \sqrt{\frac{c_2}{c_1}} \rho, \ \rho = \begin{cases} 1, & p = \infty \\ \left(\frac{2c_2}{c_3 p}\right)^{1/p}, & p \in [1, \infty) \end{cases}$$

$\mathcal{L}$ Stability of State Models

- Theorem Cont'd. *If the origin is g.e.s and all assumptions hold for globally (with $D = R^n$ and $D_u = R^m$), then for each $x_0 \in R^n$ the system is finite-gain $\mathcal{L}_p$ stable for each $p \in [1, \infty]$*
- Exercise: Provide similar conditions for finite-gain $\mathcal{L}_p$ stability of LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- **Example:**

$$\begin{aligned} \dot{x} &= -x - x^3 + u, \quad x(0) = x_0 \\ y &= tanhx + u \end{aligned}$$

  - The origin of $\dot{x} = -x - x^3$ is g.e.s. (Use Lyap $V(x) = x^2/2$)
  - $c_1 = c_2 = 1/2, \quad c_3 = c_4 = 1$
  - $L = \eta_1 = \eta_2 = 1$
  - The system is finite-gain $\mathcal{L}_p$ stable

$\mathcal{L}$ Stability of State Models

- **Example:**
$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = -x_1 - x_2 - a tanh x_1 + u, \quad a \geq 0$$
$$y = x_1$$
  - For unforced system, take $V(x) = x^T P x = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$
  - $\dot{V} = -2p_{12}(x_1^2 + ax_1 tanh x_1) + 2(p_{11} - p_{12} - p_{22})x_1x_2 - 2ap_{22}x_2 tanh x_1 - 2(p_{22} - p_{12})x_2^2$
  - To cancel the cross product term x_1x_2, choose $p_{11} = p_{12} + p_{22}$
  - To make P p.d., choose $p_{22} = 2p_{12} = 1$
  - Use the facts: $x_1 \tanh x_1 > 0$, and $\forall x_1 \in R$, $|x_1| \geq |\tanh x_1|$
  - $\rightsquigarrow \dot{V} = -x_1^2 - x_2^2 - ax_1 \tanh x_1 - 2ax_2 \tanh x_1 \leq -(1-a)\|x\|_2^2$
  - $\therefore$ for all $a < 1$, $\dot{V} < 0$
  - $c_1 = \lambda_{min}(P)$, $c_2 = \lambda_{max}(P)$, $c_3 = 1 - a$ and $c_4 = 2\|P\|_2 = 2\lambda_{max}(P)$
  - $L = \eta_1 = 1$, $\eta_2 = 0$
  - All conditions are satisfied globally $\rightsquigarrow$ system is finite-gain $\mathcal{L}_P$ stable

$\mathcal{L}$ Stability of State Models

- Theorem: *Consider the system (7) and take $r_u, r > 0$ s.t. $\{\|x\| \leq r\} \subset D$ and $\{\|u\| \leq r_u\} \subset D_u$. Suppose that*
  - *$x = 0$ is an a.s. Equu. point of $\dot{x} = f(t, x, 0)$ and there is a Lyap. fcn $V(t, x)$ and class $\mathcal{K}$ fcns α_i, $i = 1, ..., 4$ that*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|), \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(\|x\|)$$

$$\|\frac{\partial V}{\partial x}\| \leq \alpha_4(\|x\|) \qquad \forall (t,x) \in [0,\infty) \times D,$$

  - *$\forall (t, x, u) \in [0, \infty) \times D \times D_u$ and for some class $\mathcal{K}$ α_i, $i = 5, .., 7$, nonneg conts. η:*

    $\|f(t,x,u) - f(t,x,0)\| \leq \alpha_5(\|u\|)$, $\|h(t,x,u)| \leq \alpha_6(\|x\|) + \alpha_7(\|u\|) + \eta$

- *Then for each $\|x_0\| \leq \alpha_2^{-1}(\alpha_1(r))$ the system is small-signal $\mathcal{L}_\infty$ stable*

$\mathcal{L}$ Stability of State Models

- **Theorem**: *Consider the system (7) with $D = R^n$ and $D_u = R^m$. Suppose that*
  - *The system is ISS.*
  - *for all $(t, x, u) \in [0, \infty) \times R^n \times R^m$, some class $\mathcal{K}$ fcns α_1, α_2 and a const. $\eta \geq 0$*
  $$\|h(t, x, u)\| \leq \alpha_6(\|x\|) + \alpha_7(\|u\|) + \eta$$
- *Then for each $x_0 \in R^n$, the system is $\mathcal{L}_\infty$ stable.*
- **Example :**
$$\begin{aligned} \dot{x} &= -x - 2x^3 + (1 + x^2)u^2 \\ y &= x^2 + u \end{aligned}$$
  - $V = x^2/2 \rightsquigarrow \dot{V} = -x^2 - 2x^4 + x(1 + x^2)u^2 \leq -x^4 \quad \forall |x| \geq u^2$
  - $\therefore$ the system is ISS with $\gamma = r^2$
  - $\alpha_6 = r^2, \ \alpha_7 = r$ and $\eta = 0$
  - Therefore the system is $\mathcal{L}_\infty$ stable

$\mathcal{L}$ Stability of State Models

- **Example:**

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + g(t)x_2 \\ \dot{x}_2 &= -g(t)x_1 - x_2^3 + u \\ y &= x_1 + x_2 \end{aligned}$$

g(t) is continuous and bounded for $t \geq 0$

- $V = x_1^2 + x_2^2$
- $\dot{V} = -2x_1^4 - 2x_2^4 + 2x_2 u$
- $2(x_1^4 + x_2^4) \geq \|x\|_2^4$ and $0 < \theta < 1 \rightsquigarrow \dot{V} \leq -(1-\theta)\|x\|_2^4, \forall \|x\|_2 \geq (\frac{2|u|}{\theta})^{1/3}$
- globally: $\alpha_1(r) = \alpha_2(r) = r^2$, $W_3(x) = -(1-\theta)\|x\|_2^4$ and $\rho(r) = (2r/\theta)^{1/3}$
- $\therefore$ The system is globally ISS
- globally: $\alpha_6(r) = \sqrt{2}r,\ \alpha_7(r) = 0$, and $\eta = 0$
- Therefore the system is $\mathcal{L}_\infty$ stable