Computational Intelligence
Lecture 6:Fuzzy Rule Base and Fuzzy
Inference Engine

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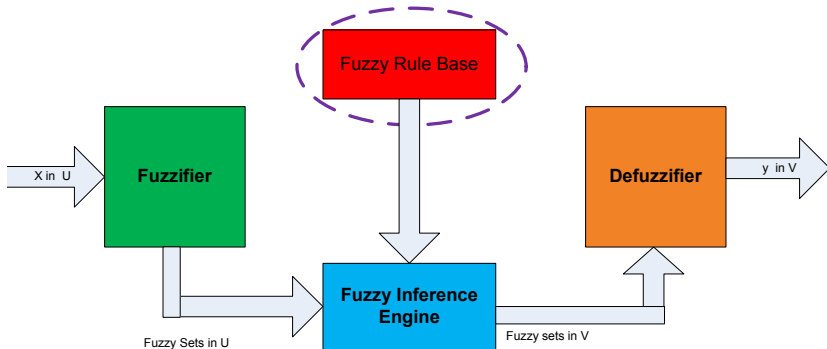Amirkabir University of Technology

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Fuzzy Rule Base

Fuzzy Rule Base

- We study only the multi-input-single-output case
- a multioutput system can be decomposed into a collection of single-output systems.
- A fuzzy rule base : a set of fuzzy IF-THEN rules.
- It is ♡ of the fuzzy system: all other components are used to implement these rules in a reasonable and efficient manner.
- The Canonical Fuzzy IF-THEN Rule
  $\text{Ru}^{(l)}$: IF x_1 is A_1^l and ... and x_n is A_n^l, THEN y is B^l
  - A_i^l: fuzzy set in $U_i \subset R$
  - B_l: Fuzzy set in $V \subset R$
  - $X = (x_1, ..., x_n)^T \in U$: Input linguistic variable
  - $y \in V$: Output linguistic variable
  - $l = 1, ..., M$, M: # of rules in the fuzzy rule base

Most of Fuzzy Rules can be expressed by Canonical Fuzzy Rule

- ▶ Partial Rules:
  IF x_1 is A_1^l and ... and x_m is A_m^l, THEN y is B^l, $m < n$
  - ▶ It is equivalent to:
    IF x_1 is A_1^l and ... and x_m is A_m^l and x_{m+1} is I and ... and x_n is I THEN y is B^l
  - ▶ I is a fuzzy set in R, $\mu_I(x) = 1, \forall x \in R$

- ▶ Or Rules:
  IF x_1 is A_1^l and ... and x_m is A_m^l or x_{m+1} is A_{m+1}^l and ... and x_n is A_n^l THEN y is B^l,
  - ▶ It is equivalent to:
    IF x_1 is A_1^l and ... and x_m is A_m^l THEN y is B^l
    IF x_{m+1} is A_{m+1}^l and ... and x_n is A_n^l THEN y is B^l

▶ Single fuzzy statement:
  y **is** B^l,

  ▶ It is equivalent to:
    IF x_1 is I and ... and x_n is I THEN y is B^l

▶ Gradual Rules:
  The smaller the x, **the bigger the** y,

  ▶ Define S : a fuzzy set for "smaller": $\mu_S(x) = \frac{1}{1+exp(5(x+2))}$
  ▶ Define B : a fuzzy set for "bigger": $\mu_B(y) = \frac{1}{1+exp(-5(y-2))}$
  ▶ The rule is equivalent to:
    IF x is S THEN y is B

▶ Non-fuzzy Rules:

 ▶ When $\mu_{A_i^l}$ and μ_{B^l} can only take values 1 or 0

Properties of Set of Rules

▶ **Complete Fuzzy Rules**

    ▶ $\forall x \in U, \exists$ at least one rule (k) in fuzzy rule base s.t. $\mu_{A_i^k} \neq 0$, for all $i = 1, ..., n$

    ▶ **Example:**

        ▶ Consider a 2-input 1-output system

        ▶ The completer Fuzzy rule set:
            IF x_1 **is** S_1 **and** x_2 **is** S_2, **THEN** y **is** B^1
            IF x_1 **is** S_1 **and** x_2 **is** L_2, **THEN** y **is** B^2
            IF x_1 **is** M_1 **and** x_2 **is** S_2, **THEN** y **is** B^3
            IF x_1 **is** M_1 **and** x_2 **is** L_2, **THEN** y **is** B^4
            IF x_1 **is** L_1 **and** x_2 **is** S_2, **THEN** y **is** B^5
            IF x_1 **is** L_1 **and** x_2 **is** L_2, **THEN** y **is** B^6

        ▶ If each of them is missing is will loose the completeness
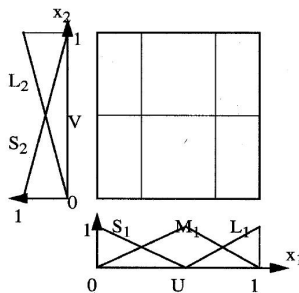
Properties of Set of Rules

- **Consistent Fuzzy Rules:**
  there are no rules with the same IF parts but different THEN parts.
  - Consistency is very important for nonfuzzy production rules, it causes conflicting rules
  - For fuzzy rules: the proper inference engine and defuzzifier use average solution to resolve the conflicting results
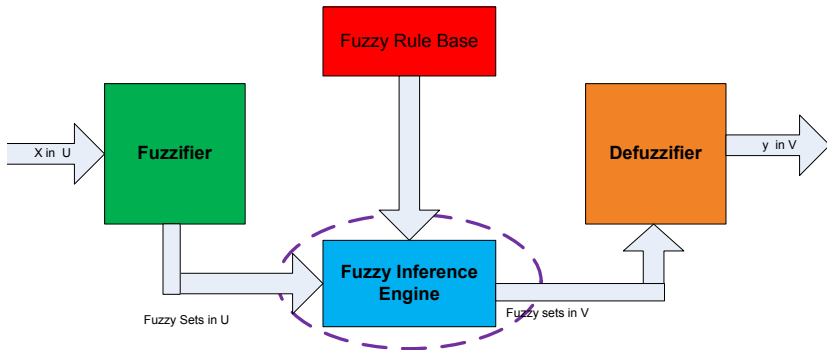
- **Continuous Fuzzy Rules:**
  there is no neighboring rules whose THEN part fuzzy sets have empty intersection.
  - ∴ I/O behavior of fuzzy system should be smooth

Fuzzy Inference

Fuzzy Inference

▶ Using fuzzy logic principles, the fuzzy IF-THEN rules in the fuzzy rule base are combined to map a fuzzy set A' in U to a fuzzy set B' in V.

▶ It is the brain of fuzzy system

▶ There are two methods to infer the rules
  1. Composition based inference
  2. Individual-rule based inference

▶ **Composition Based Inference**
  ▶ all rules in the fuzzy rule base are combined into a single fuzzy relation in $U \times V$
  ▶ It is viewed as a single fuzzy IF-THEN rule.

Composition Based Inference

▶ What is appropriate logic to combine?
▶ There are two diff. views:
  1. Each rule is independent conditional statement $\rightsquigarrow$ operator: union
     ▶ For M rules, $Ru^{(l)} = A_1^l \times \ldots \times A_n^l \to B^l$
     ▶ Interpreted as a single fuzzy relation called Mamdani combination
       $Q_M = \bigcup_{l=1}^{M} Ru^{(l)}$
     ▶ $\mu_{Q_M}(x, y) = \mu_{Ru^{(1)}}(x, y) \,\dot{+}\, \ldots \,\dot{+}\, \mu_{Ru^{(1)}}(x, y)$
       $\dot{+}$ is s-norm
  2. The rules are strongly coupled conditional statements; all the rules
     must be satisfied to have impact $\rightsquigarrow$ appropriate operator is intersection
     ▶ Combined as a fuzzy relation called Godel combination
       $\mu_{Q_G}(x, y) = \bigcap_{l=1}^{M} Ru^{(l)}$
     ▶ $\mu_{Q_G}(x, y) = \mu_{Ru^{(1)}}(x, y) \star \ldots \star \mu_{Ru^{(1)}}(x, y)$
       $\star$ is t-norm

Composition Based Inference

▶ Output of fuzzy inference engine is obtained using the generalized modus ponens:

$$\mu_{B'}(y) = sup_{x \in U} t[\mu_{A'}(x), \mu_{Q_{M/G}}(x, y)]$$

▶ **The computational procedure of the composition based inference:**

    1. Consider M canonical IF-THEN rules
      (Ru$^{(l)}$: IF x_1 is A_1^l and ... and x_n^l is A_n^l, THEN y is B^l), determine
      membership fcn: $\mu_{A_1^l} \times \ldots \times \mu_{A_n^l}(x_1, \ldots, x_n) = \mu_{A_1^l}(x_1) \star \ldots \star \mu_{A_n^l}(x_1)$
      for $l = 1, ..., M$

Composition Based Inference

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2. Considering $A_1^l \times \ldots \times A_n^l$ as $FP1$, and B^l as $FP2$, determine $\mu_{Ru^{(l)}}(x_1, \ldots, x_n, y) = \mu_{A_1^l \times \ldots \times A_n^l \to B^l}(x_1, \ldots, x_n, y)$ for $l = 1, ..., M$ using Dienes, Lukasiewicz, Zadeh, Godel, or Mamdani implications

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   for $l = 1, ..., M$

2. Considering $A_1^l \times \ldots \times A_n^l$ as $FP1$, and B^l as $FP2$, determine
   $\mu_{Ru^{(l)}}(x_1, \ldots, x_n, y) = \mu_{A_1^l \times \ldots \times A_n^l \to B^l}(x_1, \ldots, x_n, y)$ for $l = 1, ..., M$ using
   Dienes, Lukasiewicz, Zadeh, Godel, or Mamdani implications

3. Determine $\mu_{Q_G}(x, y) = \mu_{Ru^{(1)}}(x, y) \star \ldots \star \mu_{Ru^{(1)}}(x, y)$ or
   $\mu_{Q_M}(x, y) = \mu_{Ru^{(1)}}(x, y) \dot{+} \ldots \dot{+} \mu_{Ru^{(1)}}(x, y)$

Composition Based Inference

▶ Output of fuzzy inference engine is obtained using the generalized modus ponens:

$$\mu_{B'}(y) = sup_{x \in U} t[\mu_{A'}(x), \mu_{Q_{M/G}}(x, y)]$$

▶ **The computational procedure of the composition based inference:**

1. Consider M canonical IF-THEN rules
   ($Ru^{(l)}$: IF x_1 is A_1^l and ... and x_n^l is A_n^l, THEN y is B^l), determine membership fcn: $\mu_{A_1^l} \times \ldots \times \mu_{A_n^l}(x_1, \ldots, x_n) = \mu_{A_1^l}(x_1) \star \ldots \star \mu_{A_n^l}(x_1)$ for $l = 1, ..., M$

2. Considering $A_1^l \times \ldots \times A_n^l$ as $FP1$, and B^l as $FP2$, determine $\mu_{Ru^{(l)}}(x_1, \ldots, x_n, y) = \mu_{A_1^l \times \ldots \times A_n^l \to B^l}(x_1, \ldots, x_n, y)$ for $l = 1, ..., M$ using Dienes, Lukasiewicz, Zadeh, Godel, or Mamdani implications

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4. Determine output B', give input A' and $\mu_{B'}(y) = sup_{x \in U} t[\mu_{A'}(x), \mu_{Q_{M/G}}(x, y)]$

Individual-Rule Based Inference

- ▶ Each rule in the fuzzy rule base determines an output fuzzy set
- ▶ The output of the whole fuzzy inference engine is the combination of the M individual fuzzy sets using intersection or union for combination.
- ▶ **The computational procedure of the individual-rule based inference:**
    1. Follow Step 1 and 2 of composition based inference.

Individual-Rule Based Inference

▶ Each rule in the fuzzy rule base determines an output fuzzy set

▶ The output of the whole fuzzy inference engine is the combination of the M individual fuzzy sets using intersection or union for combination.

▶ **The computational procedure of the individual-rule based inference:**

  1. Follow Step 1 and 2 of composition based inference.
  2. For given input fuzzy set $A' \in U$, find output fuzzy set $B' \in V$ for each individual rule $RU^{(l)}$ according to the generalized modus ponens
     $\mu_{B'_l} = \sup_{x \in U} t[\mu_{A'}(x), \mu_{Ru^{(l)}}(x, y)]$ for $l = 1, ..., M$

Individual-Rule Based Inference

- Each rule in the fuzzy rule base determines an output fuzzy set
- The output of the whole fuzzy inference engine is the combination of the M individual fuzzy sets using intersection or union for combination.
- **The computational procedure of the individual-rule based inference:**
    1. Follow Step 1 and 2 of composition based inference.
    2. For given input fuzzy set $A' \in U$, find output fuzzy set $B' \in V$ for each individual rule $RU^{(l)}$ according to the generalized modus ponens
    $\mu_{B'_l} = \sup_{x \in U} t[\mu_{A'}(x), \mu_{Ru^{(l)}}(x, y)]$ for $l = 1, ..., M$
    3. The output of the fuzzy inference engine is the combination of the M fuzzy sets $B'_l, ..., B'_M$ either by union
    $\mu_{B'}(y) = \mu_{B'_1}(y) +\!\cdot\ldots +\!\cdot \mu_{B'_M}(y)$
    or by intersection
    $\mu_{B'}(y) = \mu_{B'_1}(y) \star \ldots \star \mu_{B'_M}(y)$

- ▶ There are a variety of choices in the fuzzy inference engine based on
  - ▶ Composition based inference or individual-rule based inference, and within the composition based inference, Mamdani inference or Godel inference
  - ▶ Dienes-Rescher implication, Lukasiewicz implication, Zadeh implication, Godel implication, or Mamdani implications
  - ▶ Different operations for the t-norms and s-norms
- ▶ **Three main criteria to choose these alternatives:**
  - ▶ Intuitive appeal: The choice should make sense intuitively. For example, if an expert who believes that the rules are independent of each other, use engine with combined by union.
  - ▶ Computational efficiency
  - ▶ Special properties: Some choice may result in an inference engine that has special properties.

A Number of Popular Fuzzy Inference Engine

- **Product Inference Engine:** Includes:
  - Individual rule based inference with union combination
  - Mamdani's product implication
  - Algebraic product for t-norms and max for all the s-norms

  $$\mu_{B'}(y) = \max_{l=1}^{M}[\sup_{x \in U}(\mu_{A'}(x) \prod_{i=1}^{n} \mu_{A_i^l}(x_i)\mu_{B^l}(y))]$$

  for given fuzzy set $A' \in U$ az input

- **Minimum Inference Engine:** Includes:
  - Individual-rule based inference with union combination
  - Mamdani's minimum implication
  - min for t-norms and max for s-norms

  $$\mu_{B'}(y) = \max_{l=1}^{M}[\sup_{x \in U} \min(\mu_{A'}(x), \mu_{A_1^l}(x_i), \ldots, \mu_{A_n^l}(x_i), \mu_{B^l}(y))]$$

  for given fuzzy set $A' \in U$ az input

- Product and Minimum inference engines are computationally simple; they are intuitively appealing for many practical problems, especially for fuzzy control.

A Number of Popular Fuzzy Inference Engine

- ▶ A disadvantage of the product and minimum inference engines: if at some $x \in U$ the $\mu_{A_i^l}(x_i)$ are very small, the obtained $\mu_{B'}(y)$ is very small. (due to local implication)
- ▶ **Lukasiewicz Inference Engine:** Includes:
    - ▶ Individual-rule based inference with intersection combination
    - ▶ Lukasiewicz implication
    - ▶ min for t-norms

  $\mu_{B'}(y) = \min_{l=1}^{M}[\sup_{x \in U} \min(\mu_{A'}(x), \mu_{Ru^{(l)}}(x, y))]$

  $\mu_{Ru^{(l)}}(x, y) = \min(1, 1 - \min_{i=1}^{n}(\mu_{A_i^l}(x_i)) + \mu_{B'}(y)))$

  $\therefore \mu_{B'}(y) = \min_{l=1}^{M}[\sup_{x \in U} \min(\mu_{A'}(x), 1 - \min_{i=1}^{n}(\mu_{A_i^l}(x_i)) + \mu_{B'}(y))]$

- **Zadeh Inference Engine:** Includes:
  - Individual rule based inference with intersection combination
  - Zadeh implication
  - min for t-norms.
    $\mu_{B'}(y) =$
    $\min_{l=1}^{M} \{ \sup_{x \in U} \min[\mu_{A'}(x), \max(\min(\mu_{A_1^l}(x_1), \ldots, \mu_{A_n^l}(x_n), \mu_B^l(y))$
    $, 1 - \min_{i=1}^{n}(\mu_{A_i^l}(x_i)))] \}$

- **Dienes-Rescher Inference Engine:** Includes:
  - Individual rule based inference with intersection combination
  - Dienes-Rescher implication
  - min for t-norms.
    $\mu_{B'}(y) = \min_{l=1}^{M} \{ \sup_{x \in U} \min[\mu_{A'}(x), \max(1 - \min_{i=1}^{n}(\mu_{A_i^l}(x_i)), \mu_B^l(y))] \}$

▶ **Lemma:** The product inference engine is unchanged if "individual rule based inference with union combination" is replaced by "composition based inference with Mamdani combination"

▶ **Proof:**
  ▶ composition based inference with Mamdani combination:
    $\mu_{B'}(y) = sup_{x \in U} t[\mu_{A'}(x), \mu_{Q_{M/G}}(x, y)]$
  ▶ $\mu_{Q_M}(x, y)$ for max as s-norm, product as t-norm:
    $\mu_{Q_{M/G}}(x, y) = \max_{l=1}^{M}(\mu_{Ru^{(l)}}(x, y))$
  ▶ $\therefore \mu_{B'}(y) = sup_{x \in U}[\mu_{A'}(x) \max_{l=1}^{M}(\mu_{Ru^{(l)}}(x, y))]$
  ▶ Use Mamdani product and
    $\mu_{A_1^l} \times \ldots \times \mu_{A_n^l}(x_1, \ldots, x_n) = \mu_{A_1^l}(x_1) \star \ldots \star \mu_{A_n^l}(x_1)$
  ▶ $\therefore \mu_{B'}(y) = sup_{x \in U} \max_{l=1}^{M}[\mu_{A'}(x) \prod_{l=1}^{n} \mu_{A_i^l}(x_i) \mu_{B'}(y)]$
  ▶ Since $\max_{l=1}^{M}$ and $sup_{x \in U}$ are interchangeable the above equation is equal to individual rule based inference with union combination

▶ If the fuzzy set A' is a fuzzy singleton: $\mu_{A'}(x) = \begin{cases} 1 & \text{if } x = x^* \\ 0 & \text{otherwise} \end{cases}$

where x^* is some point in U

- ▶ The product inference engine is $\mu_{B'}(y) = \max_{i=1}^{M}[\prod_{i=1}^{n} \mu_{A_i^l}(x_i^*)\mu_{B^l}(y)]$
- ▶ The minimum inference engine is
  $\mu_{B'}(y) = \max_{i=1}^{M}[\min(\mu_{A_1^l}(x_1^*), \ldots, \mu_{A_n^l}(x_n^*), \mu_{B^l}(y))]$
- ▶ The Lukasiewicz inference engine is
  $\mu_{B'}(y) = \min_{l=1}^{M}[1, 1 - \min_{i=1}^{n}(\mu_{A_i^l}(x_i^*)) + \mu_{B^l}(y)]$
- ▶ The Zadeh inference engine is $\mu_{B'}(y) =$
  $\min_{l=1}^{M}\{\max[\min(\mu_{A_i^l}(x_i^*), \ldots, \mu_{A_n^l}(x_n^*), \mu_B^l(y)), 1 - \min_{i=1}^{n}(\mu_{A_i^l}(x_i^*))]\}$
- ▶ The Dienes-Rescher inference engine is
  $\mu_{B'}(y) = \min_{l=1}^{M}\{\max(1 - \min_{i=1}^{n}(\mu_{A_i^l}(x_i^*)), \mu_B^l(y))\}$

▶ Graphical Minimum inference Engine
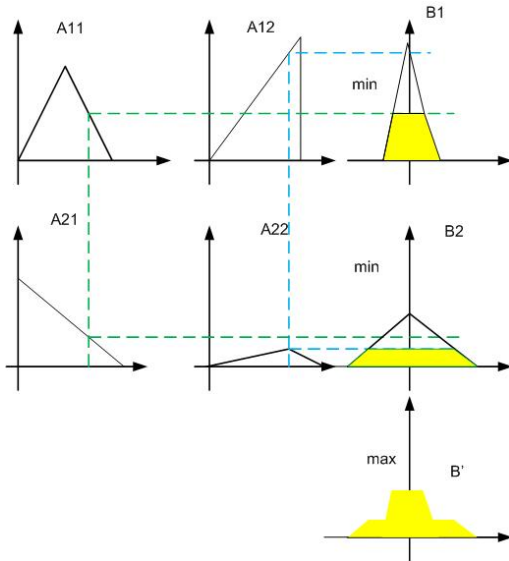
▶ Graphical Product inference Engine
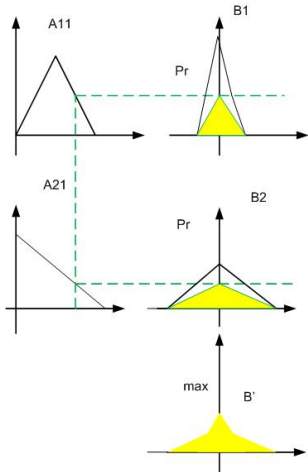
Example

- A fuzzy rule base: "IF x_1 is A_1 and ... and x_n is A_n, THEN y is B

- $\mu_B(y) = \begin{cases} 1 - |y| & \text{if } -1 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

- A' is a fuzzy singleton: $\mu_{A'}(x) = \begin{cases} 1 & \text{if } x = x^* \\ 0 & \text{otherwise} \end{cases}$

- Find $\mu_{B'}(y)$ using $B'_P, B'_M, B'_L, B'_Z, B'_D$

- Let $\min[\mu_{A_1}(x_1^*), \ldots, \mu_{A_n}(x_n^*)] = \mu_{A_p}(x_p^*)$

- $\prod_{i=1}^{n} \mu_{A_i}(x_i^*) = \mu_A(x^*)$

► $\mu_{B'_p}(y) = \mu_A(x^*)\mu_{B'}(y)$

$\mu_{A_p}(x^*) > 0.5,$ $\mu_{A_p}(x^*) \le 0.5$
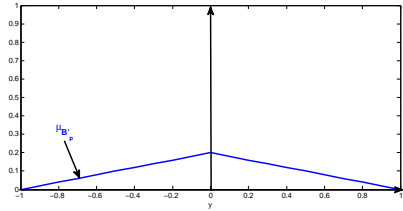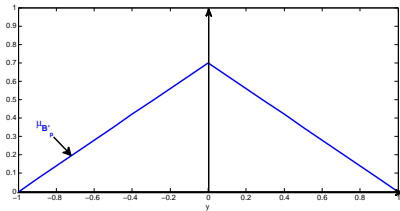
▶ $\mu_{B'_P}(y) = \mu_A(x^*)\mu_{B'}(y)$

▶ $\mu_{B'_M}(y) = \min(\mu_{A^*_p}(x^*_p), \mu_B(y))$
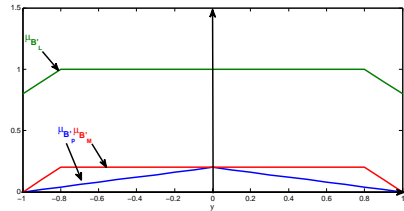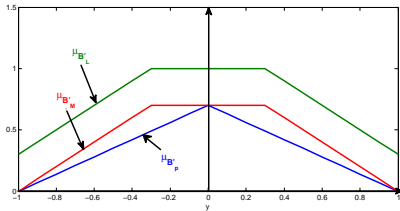
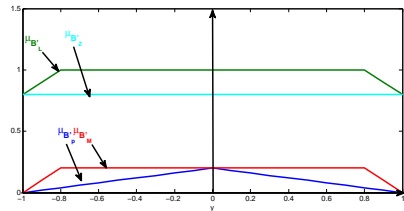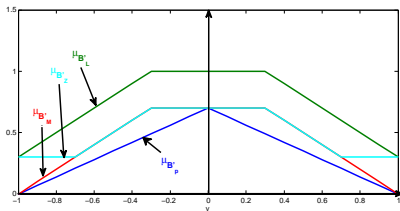$\mu_{A_p}(x^*) > 0.5,$ $\mu_{A_p}(x^*) \leq 0.5$
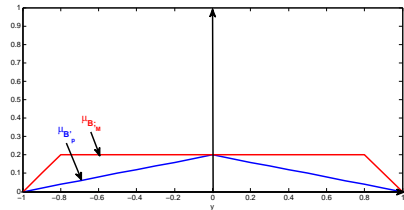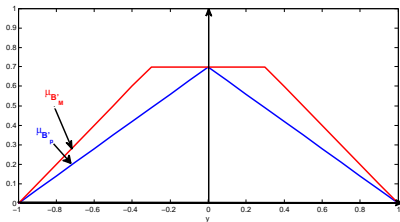
- $\mu_{B'_P}(y) = \mu_A(x^*)\mu_{B'}(y)$
- $\mu_{B'_M}(y) = \min(\mu_{A_p^*}(x_p^*), \mu_B(y))$
- $\mu_{B'L}(y) = \min[1, 1 - \mu_{A_p^*}(x_p^*) + \mu_B(y)]$

$\mu_{A_p}(x^*) > 0.5,$ $\qquad\qquad\qquad\qquad\qquad \mu_{A_p}(x^*) \leq 0.5$

- $\mu_{B'_P}(y) = \mu_A(x^*)\mu_{B'}(y)$
- $\mu_{B'_M}(y) = \min(\mu_{A^*_p}(x^*_p), \mu_B(y))$
- $\mu_{B'L}(y) = \min[1, 1 - \mu_{A^*_p}(x^*_p) + \mu_B(y)]$
- $\mu_{B'_Z}(y) = \min\{\max[\mu_{A^*_p}(x^*_p), \mu_B(y), 1 - \mu_{A^*_p}(x^*_p)]\}$
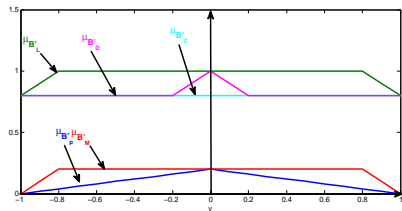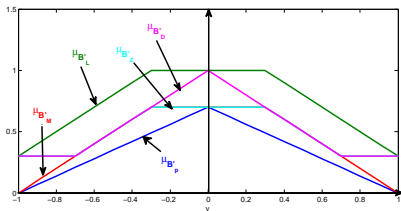
$\mu_{A_p}(x^*) > 0.5,$ $\qquad\qquad\qquad\qquad \mu_{A_p}(x^*) \leq 0.5$

- $\mu_{B'_P}(y) = \mu_A(x^*)\mu_{B'}(y)$
- $\mu_{B'_M}(y) = \min(\mu_{A_p^*}(x_p^*), \mu_B(y))$
- $\mu_{B'L}(y) = \min[1, 1 - \mu_{A_p^*}(x_p^*) + \mu_B(y)]$
- $\mu_{B'_Z}(y) = \min\{\max[\mu_{A_p^*}(x_p^*), \mu_B(y), 1 - \mu_{A_p^*}(x_p^*)]\}$
- $\mu_{B'_D}(y) = \max(1 - \mu_{A_p^*}(x_p^*), \mu_B(y))$

$\mu_{A_p}(x^*) > 0.5,$ \hspace{4cm} $\mu_{A_p}(x^*) \leq 0.5$

Example Cont'd

- $\mu_{A_p}(x^*) < 0.5 \rightsquigarrow \mu_{B'P}, \mu_{B'_M}$ is very small; $\mu_{B'L}(y), \mu_{B'_Z}(y), \mu_{B'_D}(y)$ is very large

- Product and Minimum inf. eng. are similar; Zadeh, Dienes, Lukasiewicz inf. eng. are similar

- Lukasiewicz inf. eng.: Largest output; Product inf. eng.: Smallest output