Nonlinear Control
Lecture 5: Stability Analysis II

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Comparison Functions

- Unlike autonomous systems, the solution of non-autonomous systems starting at $x(t_0) = x_0$ depends on both t and t_0.
- Stability definition shall be refined s.t. they hold uniformly in t_0.
- Two special classes of comparison functions known as class $\mathcal{K}$ and class $\mathcal{KL}$ are very useful in such definitions.
- **Definition:** *A continuous function $\alpha : [0, a) \longrightarrow [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \longrightarrow \infty$ as $r \longrightarrow \infty$.*
- **Definition:** *A continuous function $\beta : [0, a) \times [0, \infty) \longrightarrow [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belong to class $\mathcal{K}$ w.r.t. r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing w.r.t. s and $\beta(r, s) \longrightarrow 0$ as $s \longrightarrow \infty$.*

Comparison Functions

- **Examples:**
  - The function $\alpha(r) = tan^{-1}(r)$ belongs to class $\mathcal{K}$ but not to class $\mathcal{K}_\infty$.
  - The function $\alpha(r) = r^c, \ c > 0$ belongs to class $\mathcal{K}_\infty$.
  - The function $\beta(r, s) = r^c e^{-s}, \quad c > 0$ belong to class $\mathcal{KL}$.
- **Lemma:** *Let $V : D \longrightarrow R$ be a continuous positive definite function defined on a domain $D \in R^n$ containing the origin . Let $B_r \subset D$ for some $R > 0$. Then $\exists$ class $\mathcal{K}$ functions α_1 and α_2 defined on $[0, r]$ s.t.*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$\forall x \in B_r$. If $D = R^n$, the functions α_1 and α_2 will be defined on $[0, \infty)$ and the above inequality will hold for all $x \in R^n$. Moreover, if $V(x)$ is radially unbounded, then α_1 and α_2 can be chosen to belong to class $\mathcal{K}_\infty$.

- For a quadratic p.d. function $V(x) = x^T Px$, the lemma follows from the following inequality:

$$\lambda_{min}(P)\|x\|_2^2 \leq V(x) \leq \lambda_{max}(P)\|x\|_2^2$$

Nonautonomous systems

- Consider the nonautonomous system

$$\dot{x} = f(t, x) \tag{1}$$

where $f : [0, \infty) \times D \longrightarrow R^n$ is p.c. in t and locally Lip. in x on $[0, \infty) \times D$, and $D \subset R^n$ is a domain containing the origin $x = 0$.

- The origin is an **Equ. pt.** of (1), if

$$f(t, 0) = 0, \quad \forall\ t\ \geq 0$$

- A nonzero Equ. pt. or more generally nonzero solution can be transformed to $x = 0$ by proper coordinate transformation.

Nonautonomous Systems

- Suppose $x_d(\tau)$ is a solution of the system

$$\frac{dy}{d\tau} = g(\tau, y)$$

defined for all $\tau \geq a$.

- The change of variables $x = y - x_d(\tau); \quad t = \tau - a$ transform the original system into

$$\dot{x} = g(\tau, y) - \dot{x}_d(\tau) = g(t + a, x + x_d(t + a)) - \dot{x}_d(\tau) \triangleq f(t, x)$$

- Note that $\dot{x}_d(t + a) = g(t + a, x_d(t + a)), \quad \forall\ t \geq \ 0$
- Hence, $x = 0$ is an **Equ. pt.** of the transformed system
- If $x_d(t)$ is not constant, the transformed system is always nonautonomous even when the original system is autonomous, i.e. when $g(\tau, y) = g(y)$.
- Hence, a tackling problem is more difficult to solve.

Nonautonomous Systems

- The origin $x = 0$ is a stable Equ. pt. of $\dot{x} = f(t, x)$ if for each $\epsilon > 0$ and any $t_0 \geq 0$, $\exists \;\; \delta = \delta(t_0, \epsilon) \;\; \geq 0$ s.t.

$$\|x(t_0)\| < \delta \implies \|x(t)\| < \epsilon \quad \forall t \geq t_0 \tag{2}$$

- Note that $\delta = \delta(t_0, \epsilon)$ for any $t_0 \geq 0$.
- **Example:**

$$\begin{aligned} \dot{x} &= (6t \; sint \; - 2t)x \implies \\ x(t) &= x(t_0)exp\left[\int_{t_0}^{t} (6\tau \; sin\tau \; - 2\tau)d\tau\right] \\ &= x(t_0)exp\left[6sint - 6t \; cost - t^2 - 6sint_0 + 6t_0 \; cost_0 + t_0^2\right] \end{aligned}$$

- For any t_0, the term $-t^2$ is dominant $\implies$ the exp. term is bounded $\forall \; t \geq \;\; t_0 \implies$ $|x(t)| < |x(t_0)|c(t_0) \; \forall \; t \geq \; t_0$

Nonautonomous Systems

- **Definition:** The Equ. pt. $x = 0$ of $\dot{x} = f(t, x)$ is
  1. **Uniformly stable** if, for each $\epsilon < 0$, there is a $\delta = \delta(\epsilon) > 0$ independent of t_0 s.t.
  $$\|x(t_0)\| < \delta \implies \|x(t)\| < \epsilon \ \ \forall t \geq t_0$$
  2. **Asymptotically stable** if it is stable and there is a positive constant $c = c(t_0)$ such that $x(t) \to 0$ as $t \to \infty$ for all $\|x(t_0)\| < c$.
  3. **Uniformly asymptotically stable** if it is **uniformly stable** and there is a positive constant c, independent of t_0 s.t. for all $\|x(t_0)\| < c$, $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$, uniformly in t_0; that is for each $\eta > 0$, there is $T = T(\eta) > 0$ s.t.
  $$\|x(t)\| \leq \eta, \ \ \forall t \geq t_0 + T(\eta), \ \ \forall \|x(t_0)\| < c$$
  4. **Globally uniformly asymptotically stable** if it is uniformly stable, $\delta(\epsilon)$ can be chosen to satisfy $\lim_{\epsilon \longrightarrow \infty} \delta(\epsilon) = \infty$, and for each pair of positive numbers η and c, there is $T = T(\eta, c) > 0$ s.t.
  $$\|x(t)\| \leq \eta, \ \ \forall t \geq t_0 + T(\eta, c), \ \ \forall \|x(t_0)\| < c$$

Nonautonomous Systems

- Uniform properties have some desirable ability to withstand the disturbances.
- since the behavior of autonomous systems are independent of initial time t_0, all the stability propresties for autonomous systems are uniform
- **Example:**

$$\dot{x} = -\frac{x}{1+t} \implies$$

$$x(t) = x(t_0)exp\left[\int_{t_0}^{t} \frac{-1}{1+\tau} d\tau\right] = x(t_0)\frac{1+t_0}{1+t}$$

- Since $|x(t)| \leq |x(t_0)| \ \forall \ t \geq \ t_0 \implies x = 0$ is stable
- It follows that $x(t) \longrightarrow 0$ as $t \longrightarrow \infty \implies x = 0$ is **a.s.**
- However, the convergence of $x(t)$ to zero is not uniform w.r.t. t_0
- since T is not independent of t_0, i.e., larger t_0 requires more time to get close enough to the origin.

Nonautonomous Systems

- The mentioned definitions can be stated by using class $\mathcal{K}$ and class $\mathcal{KL}$ functions:
- **Lemma:** *The Equ. pt. $x = 0$ of $\dot{x} = f(t, x)$ is*
  1. **Uniformly stable** iff there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0 s.t.
  $$\|x(t)\| \leq \alpha(\|x(t_0)\|), \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c$$
  2. **Uniformly asymptotically stable** iff there exists a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0 s.t.
  $$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c \tag{3}$$
  3. **globally uniformly asymptotically stable** iff equation (3) is satisfied for any initial state $x(t_0)$.

Nonautonomous Systems

- A special class of uniform asymptotic stability arises when the class $\mathcal{KL}$ function β takes an exponential form, $\beta(r,s) = kre^{-\lambda s}$.
- **Definition:** The Equ. pt. $x = 0$ of $\dot{x} = f(t,x)$ is **exponentially stable** if there exist positive constants $c,\ k,\ \lambda$ s.t.

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall \|x(t_0)\| < c \tag{4}$$

- and is **globally exponentially stable** if equation (4) is satisfied for any initial state $x(t_0)$.
- Lyapunov theorem for autonomous system can be extended to nonautonomous systems, besides more mathematical complexity
- The extension involving uniform stability and uniform asymptotic stability is considered.
- Note that: the powerful Lasalle's theorem is not applicable for nonautonomous systems. Instead, we will introduce Balbalet's lemma.

Nonautonomous Systems

- A function $V(t,x)$ is said to be **positive semi-definite** if $V(t,x) \geq 0$
- A function $V(t,x)$ satisfying $W_1(x) \leq V(t,x)$ where $W_1(x)$ is a continuous positive definite function, is said to be **positive definite**
- A p.d. function $V(t,x)$ is said to be **radially unbounded** if $W_1(x)$ is radially unbounded.
- A function $V(t,x)$ satisfying $V(t,x) \leq W_2(x)$ where $W_2(x)$ is a continuous positive definite function, is said to be **decrescent**
- A function $V(t,x)$ is said to be **negative semi-definite** if $-V(t,x)$ is p.s.d.
- A function $V(t,x)$ is said to be **negative definite** if $-V(t,x)$ is p.d.

Lyapunov Theorem for Nonautonomous Systems

- **Theorem:**
  - **Stability:** Let $x = 0$ be an Equ. pt. for $\dot{x} = f(t, x)$ and $D \in R^n$ be a domain containing $x = 0$. Let $V : [0, \infty) \times D \longrightarrow R$ be a continuously differentiable function s.t.:
    1. V is p.d. $\equiv$ $V(x, t) \geq \alpha(\|x\|)$, α is class $\mathcal{K}$
    2. $\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x)$ is n.s.d

    then $x = 0$ is **stable**.
  - **Uniform Stability:** If, furthermore
    3. V is decrescent $\equiv$ $V(x, t) \leq \beta(\|x\|)$, β is class $\mathcal{K}$

    then the origin is **uniformly stable**.
  - **Uniform Asymptotic Stability:** If, furthermore condition 2 is strengthened by

$$\dot{V} \leq -W_3(x)$$

where W_3 is a p.d. fcn. In other word, $\dot{V}$ is n.d., then the origin is **uniformly asymptotically stable**

Lyapunov Theorem for Nonautonomous Systems

- **Theorem** (continued):
  - **Global Uniform Asymptotic Stability:** If, the conditions above are satisfied globally $\forall x \in R^n$, and
    4. V is radially unbounded $\equiv$ α is class $\mathcal{K}_\infty$

    then the origin is **globally uniformly asymptotically stable**.
  - **Exponential Stability:** If, the conditions above are satisfied with $w_i(r) = k_i r^c, \quad i = 1, ..., 3$ for some positive constants k_i & c:

$$k_1\|x\|^c \leq V(t,x) \leq k_2\|x\|^c$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3\|x\|^c, \quad \forall x \in D,$$

    then $x = 0$ is **exponentially stable**
  - Moreover, if the assumptions hold globally, then the origin is **globally exponentially stable**
    - prove it by using Comparison Lemma

Nonautonomous Systems

- **Example:** Consider $\dot{x} = -[1 + g(t)]\, x^3$

  where g(t) is cont. and $g(t) \geq 0$ for all $t \geq 0$.
  - Let $V(x) = x^2/2$, then

    $$\dot{V} = -[1 + g(t)]\, x^4 \leq -x^4, \quad \forall\, x \in R \;\&\; t \geq 0$$

  - All assumptions of the theorem are satisfied with $W_1(x) = W_2(x) = V(x)$ and $W_3(x) = x^4$. Hence, the origin is **g.u.a.s.**
- **Example:** Consider

  $$\begin{aligned} \dot{x}_1 &= -x_1 - g(t)x_2 \\ \dot{x}_2 &= -x_1 - x_2 \end{aligned}$$

  where g(t) is cont. diff, and satisfies
  $0 \leq g(t) \leq k, \quad$ and $\quad \dot{g}(t) \leq g(t), \; \forall\, t \geq 0$
  - Let $V(t,x) = x_1^2 + [1 + g(t)]\, x_2^2$

Nonautonomous Systems

- **Example** (Cont'd)

$$x_1^2 + x_2^2 \;\le\; V(t,x) \;\le\; x_1^2 + (1+k)x_2^2, \quad \forall x \in R^2$$

  - $V(t,x)$ is p.d., decrescent, and radially unbounded.

$$\dot{V} = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]\, x_2^2$$

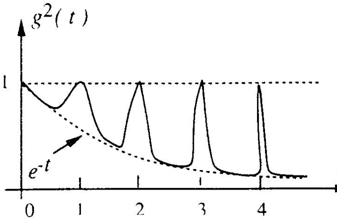
  - We have $2 + 2g(t) - \dot{g}(t) \;\ge\; 2 + 2g(t) - g(t) \;\ge\; 2.$ Then,

$$\dot{V} = -2x_1^2 + 2x_1x_2 - 2x_2^2 = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleq -x^T Q x$$

    where Q is p.d. $\Longrightarrow$ $\dot{V}(t,x)$ is n.d.
  - All assumptions of the theorem are satisfied globally with p.d. quadratic fcns W_1, W_2,, and W_3.
  - Recall: for a quadratic fcn $x^T P x$ $\lambda_{min}(P)\|x\|^2 \le x^T P x \le \lambda_{max}(P)\|x\|^2$
    The conditions of exponential stability are satisfied with $c = 2$,
    $\therefore$ origin is **g.e.s.**

Example: Importance of decrescence condition

- Consider $$\dot{x}(t) = \frac{\dot{g}(t)}{g(t)} x$$
- $g(t)$ is cont. diff fcn., coincides with $e^{-t/2}$ except around some peaks where it reaches 1. s.t.: $\int_0^\infty g^2(r)dr < \int_0^\infty e^{-r}dr + \sum_{n=1}^\infty \frac{1}{2^n} = 2$
- Let $V(x,t) = \frac{x^2}{g^2(t)}[3 - \int_0^t g^2(r)dr] \rightsquigarrow V$ is p.d ($V(x,t) > x^2$)
- $\dot{V} = -x^2$ is n.d.
- But $x(t) = \frac{g(t)}{g(t_0)} x(t_0)$
- $\therefore$ origin is not u.a.s.



The function $g^2(t)$

Linear Time-Varying Systems $\dot{x} = A(t)x$

- The sol. is the so called state transition matrix $\phi(t, t_0)$,i.e., $x(t) = \phi(t, t_0)x(t_0)$
- **Theorem:** *The Equ. pt. $x = 0$ is* **g.u.a.s** *iff the state transition matrix satisfies $\|\phi(t, t_0)\| \leq ke^{-\gamma(t-t_0)}$ $\forall t \geq t_0 > 0$ for some positive const. $k\&\gamma$*
- **u.a.s.** of $x = 0$ is equivalent to **e.s.** for linear systems.
- Tools/intutions of TI systems are **no longer valid** for TV systems.
- **Example:**

$$\ddot{x} + c(t)\dot{x} + k_0 x = 0$$

  A mass-spring-damper system with t.v. damper $c(t) \geq 0$.
- origin is an Equ. pt. of the system
- Physical intuition may suggest that the origin is **a.s.** as long as the damping $c(t)$ remains strictly positive (implying a constant dissipation of energy) as is for autonomous mass-spring-damper systems.

Linear Time-Varying Systems

- **Example** (cont'd)
  - HOWEVER, this is **not necessarily true**:

$$\ddot{x} + (2 + e^t)\dot{x} + k_0 x = 0$$

  - The sol. for $x(0) = 2, \dot{x}(0) = -1$ is $x(t) = 1 + e^{-t}$ which approaches to $x = 1$!
- **Example:** $\begin{bmatrix} -1 + 1.5cos^2 t & 1 - 1.5sint\ cost \\ -1 - 1.5sint\ cost & -1 + 1.5sin^2 t \end{bmatrix}$
  - For all t, $\lambda\{A(t)\} = -.25 \pm j.25\sqrt{7} \implies \lambda_1 \& \lambda_2$ are independent of t & lie in LHP.
  - HOWEVER, $x = 0$ is **unstable**

$$\phi(t,0) = \begin{bmatrix} e^{.5t}\ cost & e^{-t}\ sint \\ -e^{.5t}\ sint & e^{-t}\ cost \end{bmatrix}$$

Linear Time-Varying Systems

- **Important:** *For linear time-varying systems, eigenvalues of $A(t)$ cannot be used as a measure of stability.*
- A simple result: *If all eigenvalues of the symmetric matrix $A(t) + A^T(t)$ (all of which are real) remain strictly in LHP, then the LTV system is* **a.s**:

$$\exists \lambda > 0, \ \forall i, \ \forall \ t \ \geq \ 0, \quad \lambda_i\{A(t) + A^T(t)\} \ \leq \ \lambda$$

  - Consider the Lyap. fcn candidate $V = x^T x$:

$$\dot{V} = x^T \dot{x} + \dot{x}^T x = x^T \ (A(t) + A^T(t)) \ x \leq -\lambda \ x^T x = -\lambda V$$

  hence, $\forall \ t \ \geq \ 0, \quad 0 \ \leq \ x^T x = V(t) \ \leq \ V(0) e^{-\lambda t}$

  - x tends to zero exponentially. Only a **sufficient condition**, though

Linear Time-Varying Systems

- More specific theorems are available for classes of linear time-varying system such as periodic systems, slowly varying system, perturbed linear systems:
  - **Perturbed linear systems**
    - Consider a LTV system:

$$\dot{x} = (A_1 + A_2(t))x$$

    - A is sum of: A_1 is a constant Hurwitz matrix and a small t.v. matrix $A_2(t)$ satisfying:

$$A_2(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \qquad \text{and}$$

$$\int_0^\infty \|A_2(t)\| dt < \infty$$

then the LTV system is **g.e.s.**

Linear Time-Varying Systems

- **Theorem:** *Let $x = 0$ be a **e.s.** Equ. pt. of $\dot{x} = A(t)x$. Suppose, $A(t)$ is cont. & bounded. Let $Q(t)$ be a cont., bounded, p.d. and symmetric matrix, i.e $0 < c_3 I \leq Q(t) \leq c_4 I, \ \forall \ t \geq 0$. Then, there exists a cont. diff., bounded, symmetric, p.d. matrix $P(t)$ satisfying*

$$\dot{P}(t) = -(A^T(t)P(t) + P(t)A(t) + Q(t))$$

Hence, $V(t,x) = x^T P(t)x$ is a Lyap. fcn for the system satisfying e.s. theorem

  - $P(t)$ is symmetric, bounded, p.d. matrix , i.e. $0 < c_1 I \leq P(t) \leq c_2 I, \ \forall \ t \geq 0$
  - $\therefore c_1 \|x\|^2 \leq V(t,x) \leq c_2 \|x\|^2$
  - $\dot{V}(t,x) = x^T \left[\dot{P}(t) + P(t)A(t) + A^T(t)P(t)\right] x = -x^T Q(t)x \leq -c_3 \|x\|^2$
  - The conditions of exponential stability are satisfied with $c = 2 \rightsquigarrow$, the origin is **g.e.s.**

Linear Time-Varying Systems

- As a special case when $A(t) = A$, then $\phi(\tau, t) = e^{(\tau - t)A}$ which satisfies $\|\phi(t, t_0)\| \leq e^{-\gamma(t-t_0)}$ when A is a stable matrix.
- Choosing $Q = Q^T > 0$, then $P(t)$ is given by

$$P = \int_t^\infty e^{(\tau - t)A^T} \; Q \; e^{(\tau - t)A} d\tau = \int_0^\infty e^{A^T s} \; Q \; e^{As} ds$$

independent of t and is a solution to the Lyap. equation.

Linearization for Nonautonomous Systems

- Consider $\dot{x} = f(t,x)$ where $f : [0,\infty) \times D \longrightarrow R^n$ is cont. diff. and $D = \{x \in R^n \mid \|x\| < r\}$. Let $x = 0$ be an Equ. pt. Also, let the Jacobian matrix be bounded and Lip. on D uniformly in t, i.e.

$$\|\frac{\partial f}{\partial x}(t,x)\| \leq k \;\; \forall\, x \in D, \;\; \forall\, t \geq 0$$

$$\|\frac{\partial f}{\partial x}(t,x_1) - \frac{\partial f}{\partial x}(t,x_2)\| \leq L\|x_1 - x_2\| \;\; \forall\, x_1, x_2 \in D, \;\; \forall\, t \geq 0$$

Let $A(t) = \frac{\partial f}{\partial x}(t,x)\big|_{x=0}$. Then, $x = 0$ is **e.s. for the nonlinear system** if it is an **e.s. Equ. pt. for the linear system** $\dot{x} = A(t)x$.

Summary

- Lyapunov Theorem for Nonautonomous Systems $\dot{x} = f(x,t)$:
  - **Stability:** Let $x = 0$ be an Equ. pt. and $D \in R^n$ be a domain containing $x = 0$. Let $V : [0, \infty) \times D \longrightarrow R$ be a continuously differentiable function s.t.: V is p.d. , and $\dot{V}$ is n.s.d
  - **Uniform Stability:** If, furthermore V is decrescent
  - **Uniform Asymptotic Stability:** If, furthermore $\dot{V}$ is n.d.
  - **Global Uniform Asymptotic Stability:** If, the conditions above are satisfied globally $\forall x \in R^n$, and V is radially unbounded
  - **Exponential Stability:** If for some positive constants k_i & c:

$$k_1\|x\|^c \leq V(t,x) \leq k_2\|x\|^c$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3\|x\|^c, \quad \forall x \in D,$$

    then $x = 0$ is **exponentially stable**
  - Moreover, if the assumptions hold globally, then the origin is **globally exponentially stable**

Summary

- Linear Time-Varying Systems $\dot{x} = A(t)x$
  - **Theorem:** The Equ. pt. $x = 0$ is **g.u.a.s** iff the state transition matrix satisfies $\|\phi(t, t_0)\| \leq ke^{-\gamma(t-t_0)}\ \forall t \geq t_0 > 0$ for some positive const. $k \& \gamma$
  - If all eigenvalues of the symmetric matrix $A(t) + A^T(t)$ remain strictly in LHP, then the LTV system is **a.s**
  - for LTV systems, eigenvalues of $A(t)$ alone cannot be used as a measure of stability.
  - **Theorem:** Let $x = 0$ be a **e.s.** Equ. pt. of $\dot{x} = A(t)x$. Suppose, $A(t)$ is cont. & bounded. Let $Q(t)$ be a cont., bounded, p.d. and symmetric matrix, i.e $0 < c_3 I \leq Q(t) \leq c_4 I, \ \forall t \geq 0$. Then, there exists a cont. diff., bounded, symmetric, p.d. matrix $P(t)$ satisfying

$$\dot{P}(t) = -(A^T(t)P(t) + P(t)A(t) + Q(t))$$

Hence, $V(t, x) = x^T P(t)x$ is a Lyap. fcn for the system satisfying e.s. theorem

- **Converse theorems** are the inverse of Lyap. theorems.
- They guarantee the existence of Lyapunov function satisfying certain conditions, but they **do not help** in finding these fcns.
  - **Theorem:** *Let $x = 0$ be an Equ. pt. of $\dot{x} = f(t, x)$ where $f : [0, \infty) \times D \longrightarrow R^n$ is cont. diff., $D = \{x \in R^n | \ \|x\| < r\}$ and the Jacobian matrix $\frac{\partial f}{\partial x}$ is bounded on D uniformly in t. Let k, γ, and r_0 be pos constants with $r_0 < r/k$. Let $D_0 = \{x \in R^n \mid \|x\| < r_0\}$. Assume that the trajectories satisfy $\|x(t)\| \leq k \|x(t_0)\| e^{-\gamma(t-t_0)}, \ \forall\, x(t_0) \in D_0, \forall\, t \geq t_0 \geq 0$. Then, $\exists$ a fcn $V : [0, \infty) \times D_0 \longrightarrow R$ satisfying:*

$$\begin{aligned} c_1\|x\|^2 \leq V(t,x) &\leq c_2\|x\|^2 \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &\leq -c_3\|x\|^2 \\ \frac{\partial V}{\partial x} &\leq c_4\|x\| \end{aligned}$$

for some pos., const. $c_1, \ldots, c_4$. Moreover, if $r = \infty$ and the origin is **g.e.s.**, *then $V(t,x)$ is defined and satisfies the the above inequalities on R^n. If $f(t,x) = f(x)$, then $V(t,x) = V(x)$.*

Converse Theorems

- Now, exponential stability of the linearization is a necessary and sufficient condition for **e.s.** of $x = 0$
- **Theorem:** *Let $x = 0$ be an Equ. pt. of $\dot{x} = f(t, x)$ with conditions as above. Let $A(t) = \frac{\partial f(t,x)}{\partial x}\Big|_{x=0}$. Then, $x = 0$ is an* **e.s.** *Equ. pt. for the nonlinear system* **iff** *it is an* **e.s.** *Equ. pt. for the linear system $\dot{x} = A(t)x$.*
- For autonomous systems e.s. condition is satisfied iff A is Hurwitz.
- **Example:**

$$\dot{x} = -x^3$$

  - Recall that $x = 0$ is **a.s.**
  - However, linearization results in $\dot{x} = 0$ whose A is not Hurwitz.
  - Using the above theorem, we conclude that $x = 0$ is **not exponentially stable** for nonlinear system.

Barbalat's Lemma

- For autonomous systems, invariant set theorems are power tools to study asymptotic stability when $\dot{V}$ is **n.s.d.**
- The invariant set theorem is not valid for nonautonomous systems.
- Hence, *asymptotic stability* of nonautonomous systems is generally **more difficult** than that of autonomous systems.
- An important result that remedy the situation: **Barbalat's Lemma**
- **Asymptotic Properties of Functions and Their Derivatives:**
- For diff. fcn f of time t, always keep in mond the following three facts!

1. $\dot{f} \longrightarrow 0 \quad \not\Rightarrow \quad f$ converges
    - The fact that $\dot{f} \longrightarrow 0$ does not imply $f(t)$ has a limit as $t \longrightarrow \infty$.
    - **Example:** $f(t) = sin(ln\ t) \rightsquigarrow \dot{f} = \frac{cos(ln\ t)}{t} \longrightarrow 0$ as $t \longrightarrow \infty$
    - However, the fcn $f(t)$ keeps oscillating (slower and slower).
    - **Example:** For an unbounded function $f(t) = \sqrt{t}\, sin(ln\ t), \rightsquigarrow \dot{f} = \frac{sin(ln\ t)}{2\sqrt{t}} + \frac{cos(ln\ t)}{\sqrt{t}} \longrightarrow 0$ as $t \longrightarrow \infty$

Asymptotic Properties of Functions and Their Derivatives:

2. f converges $\nRightarrow$ $\dot{f} \longrightarrow 0$
   - The fact that f(t) has a finite limit at $t \longrightarrow \infty$ does not imply that $\dot{f} \longrightarrow 0$.
   - **Example:** $f(t) = e^{-t}\, sin(e^{2t}) \longrightarrow 0$ as $t \longrightarrow \infty$
   - while its derivative $\dot{f} = -e^{-t}\, sin(e^{2t}) + 2e^{t}\, cos(e^{2t})$ is unbounded.
3. If f is lower bounded and decreasing ($\dot{f} \leq 0$), then it converges to a limit.
   - However, it does not say whether the slope of the curve will diminish or not.

Given that a fcn tends towards a finite limit, what additional property guarantees that the derivatives converges to zero?

Barbalat's Lemma

- **Barbalat's Lemma:** *If the differentiable fcn has a finite limit as $t \longrightarrow \infty$, and if $\dot{f}$ is uniformly cont., then $\dot{f} \longrightarrow 0$ as $t \longrightarrow \infty$.*
  - proved by contradiction
- A function $g(t)$ is **continuous** on $[0, \infty)$ if

$$\forall\ t_1 \geq 0,\ \forall\ R > 0,\ \exists\ \eta(R, t_1) > 0,\ \forall\ t \geq 0,\ \ |t - t_1| < \eta \implies |g(t) - g(t_1)| < R$$

- A function $g(t)$ is **uniformly continuous** on $[0, \infty)$ if

$$\forall\ R > 0,\ \exists\ \eta(R) > 0,\ \forall\ t_1 \geq 0,\ \forall\ t \geq 0,\ \ |t - t_1| < \eta \implies |g(t) - g(t_1)| < R$$

i.e. an η can be found independent of specific point t_1.

Barbalat's Lemma

- A **sufficient condition** for a diff. fcn. to be **uniformly continuous** is that its **derivative** be **bounded**
  - This can be seen from Mean-Value Theorem:

$$\forall\ t,\quad \forall t_1,\quad \exists\ t_2 \text{ (between } t \text{ and } t_1) \text{ s.t. } \ g(t) - g(t_1) = \dot{g}(t_2)(t - t_1)$$

  - if $\dot{g} \ \leq \ R_1 \ \forall\ t \ \geq \ 0$, let $\eta = \frac{R}{R_1}$ independent of t_1 to verify the definition above.
- $\therefore$ An immediate and practical corollary of Barbalat's lemma: If the differentiable fcn $f(t)$ has a finite limit as $t \ \longrightarrow \ \infty$ and $\ddot{f}$ exists and is bounded, then $\dot{f} \ \longrightarrow \ 0$ as $t \ \longrightarrow \ \infty$

Using Barbalat's lemma for Stability Analysis

- **Lyapunov-Like Lemma:** *If a scaler function $V(t,x)$ satisfies the following conditions*
  1. *$V(t,x)$ is lower bounded*
  2. *$\dot{V}(t,x)$ is negative semi-definite*
  3. *$\dot{V}(t,x)$ is uniformly continuous in time*

  then $\dot{V}(t,x) \longrightarrow 0$ as $t \longrightarrow \infty$.
- **Example:**
  - Consider a simple adaptive control systems:

$$\begin{aligned} \dot{e} &= -e + \theta w(t) \\ \dot{\theta} &= -e w(t) \end{aligned}$$

where e is the tracking error, θ is the parameter error, and $w(t)$ is a bounded continuous fcn.

Example (Cont'd)

- Consider the lower bounded fcn:

$$\begin{aligned} V &= e^2 + \theta^2 \\ \dot{V} &= 2e(-e + \theta w) + 2\theta(-ew(t)) = -2e^2 \ \leq \ 0 \end{aligned}$$

- $\therefore V(t) \ \leq \ V(0)$, therefore, e and θ are bounded.
- Invariant set theorem cannot be used to conclude the convergence of e, since the dynamic is nonautonomous.
- To use Barbalat's lemma, check the uniform continuity of $\dot{V}$.

$$\ddot{V} = -4e(-e + \theta w)$$

- $\ddot{V}$ is bounded, since w is bounded by assumption and e and θ are shown to be bounded $\rightsquigarrow$ $\dot{V}$ is uniformly continuous
- Applying Barbalat's lemma: $\dot{V} = 0 \implies e \ \longrightarrow \ 0$ as $t \ \longrightarrow \ \infty$.
- **Important:** Although $e \ \longrightarrow \ 0$, the system is not **a.s.** since θ is only shown to be bounded.

Boundedness and Ultimate Boundedness

- Lyapunov analysis can be used to show boundedness of the solution even when there is no Equ. pt.
- **Example:**

$$\dot{x} = -x + \delta \, sint, \quad x(t_0) = a, \quad a > \delta > 0$$

which has no Equ. pt and

$$x(t) = e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} sin\tau \, d\tau$$

$$x(t) \leq e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} d\tau$$

$$= e^{-(t-t_0)}a + \delta \left[1 - e^{-(t-t_0)}\right] \leq a, \quad \forall \, t \geq t_0$$

  - The solution is bounded for all $t \geq t_0$, uniformly in t_0.
  - The bound is conservative due to the exponentially decaying terms.

Example Cont'd

- Pick any number b s.t. $\delta < b\ a$, it can be seen that
$$|x(t)| \leq b,\ \delta\ \forall\ t \geq t_0 + ln\left(\frac{a-\delta}{b-\delta}\right)$$
- b is also independent of t_0
  - The solution is said to be **uniformly ultimately bounded**
  - b is called the **ultimate bound**
- The same properties can be obtained via Lyap. analysis. Let $V = x^2/2$
$$\dot{V} = x\dot{x} = -x^2 + x\delta\ sint \leq -x^2 + \delta|x|$$
- $\dot{V}$ is not **n.d.**, bc. near the origin, positive linear term $\delta|x|$ is dominant.
- However, $\dot{V}$ is negative outside the set $\{|x| \leq \delta\}$.
- Choose, $c > \delta^2/2$, solutions starting in the set $\{V(x) < c\}$ will remain there in for all future time since $\dot{V}$ is negative on the boundary $V = c$.
- Hence, the solution is **ultimately bounded**.

Boundedness and Ultimate Boundedness

- **Definition:** The solutions of $\dot{x} = f(t, x)$ where $f : (0, \infty) \times D \to R^n$ is piecewise continuous in t and locally Lipschitz in x on $(0, \infty) \times D$, and $D \in R^n$ is a domain that contains the origin are
  - **uniformly bounded** if there exist a positive constant c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, independent of t_0, s.t.

$$\|x(t_0)\| \leq a \implies \|x(t)\| \leq \beta, \ \forall\ t \geq t_0 \tag{5}$$

  - **uniformly ultimately bounded** if there exist positive constants b and c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $T = T(a, b) > 0$, independent of t_0, s.t.

$$\|x(t_0)\| \leq a \implies \|x(t)\| \leq \beta, \ \forall\ t \geq t_0 + T \tag{6}$$

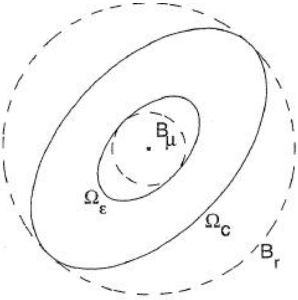
  - **globally uniformly bounded** if (5) holds for arbitrary large a.
  - **globally uniformly ultimately bounded** if (6) holds for arbitrary large a.

How to Find Ultimate Bound?

- In many problems, negative definiteness of $\dot{V}$ is guaranteed by using norm inequalities, i.e.:

$$\dot{V}(t,x) \leq -W_3(x), \quad \forall\, \mu \leq \|x\| \leq r,\ \forall\, t \geq t_0 \quad (7)$$

- If r is sufficiently larger than μ, then c and ϵ can be found s.t. the set $\Lambda = \{\epsilon \leq V \leq c\}$ is nonempty and contained in $\{\mu \leq \|x\| \leq r\}$.
- If the Lyap. fcn satisfy: $\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$ for some class $\mathcal{K}$ functions α_1 and α_2.
- We are looking for a bound on $\|x\|$ based on α_1 and α_2 s.t. satisfies (7).



How to Find Ultimate Bound?

- We have: $V(x) \leq c \implies \alpha_1(\|x\|) \leq c \iff \|x\| \leq \alpha_1^{-1}(c)$
  $\therefore\ c = \alpha_1(r)$ ensures that $\Omega_c \subset B_r$.
- On the other hand we have:
$$\|x\| \leq \mu \implies V(x) \leq \alpha_2(\mu)$$
- Hence, taking $\epsilon = \alpha_2(\mu)$ ensures that $B_\mu \subset \Omega_\epsilon$.
- To obtain $\epsilon < c$, we must have $\mu < \alpha_2^{-1}(\alpha_1(r))$.
- Hence, all trajectories starting in Ω_c enter Ω_ϵ within a finite time T as discussed before.
- To obtain the ultimate bound on $x(t)$,
$$V(x) \leq \epsilon \implies \alpha_1(\|x\|) \leq \epsilon \Leftrightarrow \|x\| \leq \alpha_1^{-1}(\epsilon)$$
- Recall that $\epsilon = \alpha_2(\mu)$, hence: $x \in \Omega_\epsilon \implies \|x\| \leq \alpha_1^{-1}(\alpha_2(\mu))$
  $\therefore$ The ultimate bound can be taken as $b = \alpha_1^{-1}(\alpha_2(\mu))$.

Ultimate Boundedness

- **Theorem:** *Let $D \in R^n$ be a domain containing the origin and $V : [0, \infty) \times D \longrightarrow R$ be a cont. diff. fcn s.t.*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) \quad (8)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x), \quad \forall \ \|x\| \geq \mu > 0 \quad (9)$$

$\forall t \geq 0$ and $\forall x$ in D, where α_1 and α_2 are class $\mathcal{K}$ fcns and $W_3(x)$ is a cont. p.d. fcn. Take $r > 0$ s.t. $B_r \subset D$ and suppose that

$$\mu < \alpha_2^{-1}(\alpha_1(r))$$

Then, there exists a class $\mathcal{KL}$ fcn β and for every initial state $x(t_0)$ satisfying $\|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$, there is $T \geq 0$ (dependent on $x(t_0)$ and μ s.t.)

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall \ t_0 \leq t \leq t_0 + T$$

$$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\mu)) \quad \forall \ t \geq t_0 + T$$

Moreover, if $D = R^n$ and $\alpha_1 \in \mathcal{K}_\infty$, then the above inequalities hold for any initial state $x(t_0)$

Ultimate Boundedness

- The inequalities of the theorem show that
  - $x(t)$ is uniformly bounded for all $t \geq t_0$
  - uniformly ultimately bounded with the ultimate bound $\alpha_1^{-1}(\alpha_2(\mu))$.
  - The ultimate function is a class $\mathcal{K}$ fcn of $\mu \rightsquigarrow$ the smaller the value of μ, the smaller the ultimate bound
  - As $\mu \longrightarrow 0$, the ultimate bound approaches zero.
- The main application of this theorem arises in studying the stability of **perturbed system**.

Example for Ultimate Boundedness

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(1+x_1^2)x_1 - x_2 + M\cos\omega t\end{aligned}$$

where $M > 0$

- Let $V(x) = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \frac{1}{2}x_1^4 = x^T P x + \frac{1}{2}x_1^4$
- $V(x)$ is p.d. and radially unbounded $\rightsquigarrow$ there exist class $\mathcal{K}_\infty$ fcns. α_1 and α_2 satisfying (8).
- $\dot{V} = -x_1^2 - x_1^4 - x_2^2 + (x_1 + 2x_2)M\cos\omega t \leq -\|x\|_2^2 - x_1^4 + M\sqrt{5}\|x\|_2$
  - where $(x_1 + 2x_2) = [1 \;\; 2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \sqrt{5}\|x\|_2$
- We want to use part of $-\|x\|_2^2$ to dominate $M\sqrt{5}\|x\|_2$ for large $\|x\|$
- $\dot{V} \leq -(1-\theta)\|x\|_2^2 - x_1^4 - \theta\|x\|_2^2 + M\sqrt{5}\|x\|_2$, for $0 < \theta < 1$
- Then $\dot{V} \leq -(1-\theta)\|x\|_2^2 - x_1^4 \quad \forall \; \|x\|_2 \geq \frac{M\sqrt{5}}{\theta} \rightsquigarrow \mu = M\sqrt{5}/\theta$

Example for Ultimate Boundedness

- $\therefore$ the solutions are u.u.b.
- Next step: finding ultimate bound:

$$V(x) \geq x^T P x \geq \lambda_{min}(P)\|x\|_2^2$$
$$V(x) \leq x^T P x + \frac{1}{2}\|x\|_2^4 \leq \lambda_{max}(P)\|x\|_2^2 + \frac{1}{2}\|x\|_2^4$$

- $\alpha_1(r) = \lambda_{min}(P)r^2$ and $\alpha_2(r) = \lambda_{max}(P)r^2 + \frac{1}{2}r^4$
- $\therefore$ ultimate bound: $b = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\lambda_{max}(P)\mu^2 + \mu^4/2\lambda_{min}(P)}$