Signals and Systems
Lecture 4: Fourier Setries(DT)

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Fourier Series for DT

- It is defined similar to CT fourier series
- But DT fourier series of a signal is linear combination of finite terms.
  - Remember harmonic sets in DT: $\phi_N[n] = e^{jN\omega_0 n} = \phi_0[n]$, N is fundamental period
- Fourier series in DT:

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

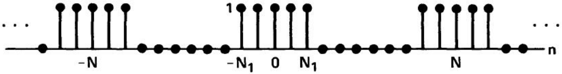
$< N >$ means N sequential samples.

- All periodic signals can be represented by fourier series.
- Since number of terms are finite, convergence study is not required

Fourier Series Coefficients

- Fourier series: $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$
- Multiply both sides by $e^{-j\omega_0 mn}$: $e^{-j\omega_0 mn} x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} e^{-j\omega_0 mn}$
- Take summation on $n =< N >$:
  $\sum_{n=<N>} e^{-j\omega_0 mn} x[n] = \sum_{n=<N>} \sum_{k=<N>} a_k e^{jk\omega_0 n} e^{-j\omega_0 mn}$
- $\sum_{n=<N>} e^{-j\omega_0 mn} x[n] = \sum_{k=<N>} a_k \sum_{n=<N>} e^{j(k-m)\omega_0 n}$
- Use the sum of geometrical series rule: $\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & \alpha = 1 \\ \frac{1-\alpha^N}{1-\alpha} & \alpha \neq 1 \end{cases}$
- $\therefore \sum_{n=<N>} e^{-j\omega_0 mn} x[n] = \sum_{k=<N>} a_k N\delta[k-m] = Na_m$
- $a_m = \frac{1}{N} \sum_{n=<N>} x[n] e^{-j\omega_0 mn}$
- $a_k = a_{k+N}$

Example: Periodic square wave



- $a_0 = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n] = \frac{2N_1+1}{N} = a_N = a_{-N} = \dots$
- for $k \neq nN, \quad m = n + N_1$

$$
\begin{aligned}
a_k &= \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\omega_0 (m-N_1)} \\
&= \frac{1}{N} e^{-jk\omega_0 N_1} \sum_{m=0}^{2N_1} e^{-jk\omega_0 m} = \frac{1}{N} e^{-jk\omega_0 N_1} \frac{1-e^{-j\omega_0(2N_1+1)}}{1-e^{jk\omega_0}} \\
&= \frac{1}{N}\frac{sin[k(N_1+\frac{1}{2})\omega_0]}{sin(k\omega_0/2)}
\end{aligned}
$$

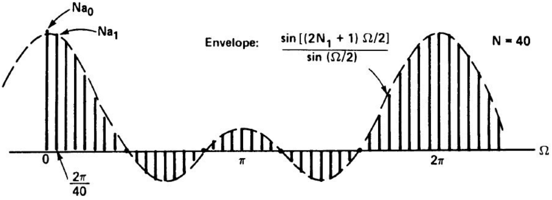


Figure : DT FS coefficients for $N = 40$

- There is no Gibbs phenomenon for DT Fourier Series
- The sum of DT FS is finite, i.e., any DT periodic sequence of $x[n]$ is completely described by a finite number of N paramters
- No approximate is made so no convergence issue

Properties of DT Fourier Series

- Properties of DT FS is very similar to CT FS
- The details are given in the table of Ref book
- Some of them with significant difference are listed here
- Consider $x[n]$ and $y[n]$: periodic signals with same fundamental period N
- $x[n] \Leftrightarrow a_k, y[n] \Leftrightarrow b_k$
- Multiplication: $w[n] = x[n]y[n] \Leftrightarrow d_k = \sum_{l=<N>} a_l b_{k-l}$
  - The result is a periodic convolution between the FS sequences
  - $w[n]$ is periodic with N
- First Difference: $x[n] - x[n-1] \Leftrightarrow (1 - e^{-jk(2\pi/N)})a_k$
  - It is parallel to first differentiation in CT FS
- Parseval's Relation: $\frac{1}{N} \sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$
  - The average power in a periodic signal = the sum of the average power in all of its harmonic components

Fourier Series and LTI Systems

- For CT e^{st} is an eigenfunction for LTI systems:
  - $x(t) = e^{st} \rightarrow y(t) = H(s)e^{st}$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

  where $h(t)$ is impulse response
- For DT z^n is an eigne function for LTI systems:
  - $x[n] = z^n \rightarrow y[n] = H(z)z^n$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

  where $h[n]$ is impulse response
- When s and z are complex numbers $H(s)$ and $H(z)$ are called system function

- If $s = j\omega$/ $z = e^{j\omega}$, $H(j\omega)/H(e^{j\omega})$ is called frequency response

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \\ H(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} h[k]z^{-j\omega k} \end{aligned} \quad (1)$$

- Now assume $x(t)$ is a periodic signal $\rightsquigarrow$ $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
- $y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$
  - $|a_k| \rightarrow |a_k||H(jk\omega_0)|$(FS coefficient just multiplied by a gain)
  - $y(t)$ is periodic with the same period
  - Effect of LTI system is modifying each Fourier coefficient of input signal by value of frequency response at that frequency
  - The same story is true for DT
- $H(j\omega)/H(e^{j\omega})$ is "well defined" if it is finite
  - Stable LTI systems has well defined freq. response