Nonlinear Control
Lecture 5: Stability Analysis II

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Nonautonomous systems

Lyapunov Theorem for Nonautonomous Systems

Linear Time-Varying Systems and Linearization

Linear Time-Varying Systems
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Barbalat’s Lemma and Lyapunov-Like Lemma

Asymptotic Properties of Functions and Their Derivatives:
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Boundedness and Ultimate Boundedness
Consider the nonautonomous system

\[ \dot{x} = f(t, x) \]  \hspace{1cm} (1)

where \( f : [0, \infty) \times D \rightarrow R^n \) is p.c. in \( t \) and locally Lip. in \( x \) on \([0, \infty) \times D\), and \( D \subset R^n \) is a domain containing the origin \( x = 0 \).

The origin is an **Equ. pt.** of (1), at \( t = 0 \) if

\[ f(t, 0) = 0, \quad \forall \ t \geq 0 \]

A nonzero Equ. pt. or more generally nonzero solution can be transformed to \( x = 0 \) by proper coordinate transformation.
Nonautonomous Systems

- Suppose $x_d(\tau)$ is a solution of the system

\[
\frac{dy}{d\tau} = g(\tau, y)
\]

defined for all $\tau \geq a$.

- The change of variables $x = y - x_d(\tau); \quad t = \tau - a$ transform the original system into

\[
\dot{x} = g(\tau, y) - \dot{x}_d(\tau) = g(t + a, x + x_d(t + a)) - \dot{x}_d(\tau) \triangleq f(t, x)
\]

- Note that $\dot{x}_d(t + a) = g(t + a, x_d(t + a)), \quad \forall \ t \geq 0$

- Hence, $x = 0$ is an Equ. pt. of the transformed system

- If $x_d(t)$ is not constant, the transformed system is always nonautonomous even when the original system is autonomous, i.e. when $g(\tau, y) = g(y)$.

- Hence, a tackling problem is more difficult to solve.
Nonautonomous Systems

The origin $x = 0$ is a stable Equ. pt. of $\dot{x} = f(t, x)$ if for each $\epsilon > 0$ and any $t_0 \geq 0$, $\exists \delta = \delta(t_0, \epsilon) \geq 0$ s.t.

$$\|x(t_0)\| < \delta \implies \|x(t)\| < \epsilon \quad \forall t \geq t_0$$  \hspace{1cm} (2)

Note that $\delta = \delta(t_0, \epsilon)$ for any $t_0 \geq 0$.

Example:

$$\dot{x} = (6t \sin t - 2t)x \implies x(t) = x(t_0)\exp \left[ \int_{t_0}^{t} (6\tau \sin \tau - 2\tau) d\tau \right]$$

$$= x(t_0)\exp \left[ 6\sin t - 6t \cos t - t^2 - 6\sin t_0 + 6t_0 \cos t_0 + t_0^2 \right]$$

For any $t_0$, the term $-t^2$ is dominant $\implies$ the exp. term is bounded

$$\forall t \geq t_0 \implies |x(t)| < |x(t_0)|c(t_0) \quad \forall t \geq t_0$$
Nonautonomous Systems

**Definition:** The Equ. pt. $x = 0$ of $\dot{x} = f(t, x)$ is

1. **Uniformly stable** if, for each $\epsilon < 0$, there is a $\delta = \delta(\epsilon) > 0$ independent of $t_0$ s.t.
   \[
   \|x(t_0)\| < \delta \implies \|x(t)\| < \epsilon \quad \forall t \geq t_0
   \]

2. **Asymptotically stable** if it is stable and there is a positive constant $c = c(t_0)$ such that $x(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $\|x(t_0)\| < c$.

3. **Uniformly asymptotically stable** if it is uniformly stable and there is a positive constant $c$, independent of $t_0$ s.t. for all $\|x(t_0)\| < c$, $x(t) \rightarrow 0$ as $t \rightarrow \infty$, uniformly in $t_0$; that is for each $\eta > 0$, there is $T = T(\eta) > 0$ s.t.
   \[
   \|x(t)\| \leq \eta, \quad \forall t \geq t_0 + T(\eta), \quad \forall \|x(t_0)\| < c
   \]

4. **Globally uniformly asymptotically stable** if it is uniformly stable, $\delta(\epsilon)$ can be chosen to satisfy $\lim_{\epsilon \rightarrow \infty} \delta(\epsilon) = \infty$, and for each pair of positive numbers $\eta$ and $c$, there is $T = T(\eta, c) > 0$ s.t.
   \[
   \|x(t)\| \leq \eta, \quad \forall t \geq t_0 + T(\eta, c), \quad \forall \|x(t_0)\| < c
   \]
Nonautonomous Systems

- Uniform properties have some desirable ability to withstand the disturbances.

- since the behavior of autonomous systems are independent of initial time $t_0$, all the stability properties for autonomous systems are uniform

- **Example:**
  \[
  \dot{x} = -\frac{x}{1 + t} \implies x(t) = x(t_0) \exp \left[ \int_{t_0}^{t} \frac{-1}{1 + \tau} d\tau \right] = x(t_0) \frac{1 + t_0}{1 + t}
  \]

  - Since $|x(t)| \leq |x(t_0)| \ \forall \ t \geq t_0 \implies x = 0$ is stable
  - It follows that $x(t) \to 0$ as $t \to \infty \implies x = 0$ is a.s.
  - However, the convergence of $x(t)$ to zero is not uniform w.r.t. $t_0$
  - since $T$ is not independent of $t_0$, i.e., larger $t_0$ requires more time to get close enough to the origin.
Unlike autonomous systems, the solution of non-autonomous systems starting at \( x(t_0) = x_0 \) depends on both \( t \) and \( t_0 \).

Stability definition shall be refined s.t. they hold uniformly in \( t_0 \).

Two special classes of comparison functions known as class \( \mathcal{K} \) and class \( \mathcal{KL} \) are very useful in such definitions.

**Definition:** A continuous function \( \alpha : [0, a) \to [0, \infty) \) is said to belong to class \( \mathcal{K} \) if it is strictly increasing and \( \alpha(0) = 0 \). It is said to belong to class \( \mathcal{K}\infty \) if \( a = \infty \) and \( \alpha(r) \to \infty \) as \( r \to \infty \).

**Definition:** A continuous function \( \beta : [0, a) \times [0, \infty) \to [0, \infty) \) is said to belong to class \( \mathcal{KL} \) if, for each fixed \( s \), the mapping \( \beta(r, s) \) belong to class \( \mathcal{K} \) w.r.t. \( r \) and, for each fixed \( r \), the mapping \( \beta(r, s) \) is decreasing w.r.t. \( s \) and \( \beta(r, s) \to 0 \) as \( s \to \infty \).
Examples:

- The function $\alpha(r) = \tan^{-1}(r)$ belongs to class $\mathcal{K}$ but not to class $\mathcal{K}_\infty$.
- The function $\alpha(r) = r^c, \ c > 0$ belongs to class $\mathcal{K}_\infty$.
- The function $\beta(r, s) = r^c e^{-s}, \ c > 0$ belong to class $\mathcal{KL}$. 
The mentioned definitions can be stated by using class $\mathcal{K}$ and class $\mathcal{KL}$ functions:

**Lemma:** The Equ. pt. $x = 0$ of $\dot{x} = f(t, x)$ is

1. **Uniformly stable** iff there exist a class $\mathcal{K}$ function $\alpha$ and a positive constant $c$, independent of $t_0$ s.t.

   \[
   \|x(t)\| \leq \alpha(\|x(t_0)\|), \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c
   \]

2. **Uniformly asymptotically stable** iff there exists a class $\mathcal{KL}$ function $\beta$ and a positive constant $c$, independent of $t_0$ s.t.

   \[
   \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c
   \] (3)

3. **globally uniformly asymptotically stable** iff equation (3) is satisfied for any initial state $x(t_0)$. 
Nonautonomous Systems

- A special class of uniform asymptotic stability arises when the class $\mathcal{KL}$ function $\beta$ takes an exponential form, $\beta(r, s) = kre^{-\lambda s}$.

- **Definition:** The Equ. pt. $x = 0$ of $\dot{x} = f(t, x)$ is exponentially stable if there exist positive constants $c$, $k$, $\lambda$ s.t.

\[
\|x(t)\| \leq k\|x(t_0)\| e^{-\lambda(t-t_0)} , \quad \forall\|x(t_0)\| < c
\]  

(4)

- and is **globally exponentially stable** if equation (4) is satisfied for any initial state $x(t_0)$.

- Lyapunov theorem for autonomous system can be extended to nonautonomous systems, besides more mathematical complexity

- The extension involving uniform stability and uniform asymptotic stability is considered.

- **Note that:** the powerful Lasalle’s theorem is not applicable for nonautonomous systems. Instead, we will introduce Balbalet’s lemma.
Nonautonomous Systems

- A function $V(t, x)$ is said to be **positive semi-definite** if $V(t, x) \geq 0$.

- A function $V(t, x)$ satisfying $W_1(x) \leq V(t, x)$ where $W_1(x)$ is a continuous positive definite function, is said to be **positive definite**.

- A p.d. function $V(t, x)$ is said to be **radially unbounded** if $W_1(x)$ is radially unbounded.

- A function $V(t, x)$ satisfying $V(t, x) \leq W_2(x)$ where $W_2(x)$ is a continuous positive definite function, is said to be **decrrescent**.

- A function $V(t, x)$ is said to be **negative semi-definite** if $-V(t, x)$ is p.s.d.

- A function $V(t, x)$ is said to be **negative definite** if $-V(t, x)$ is p.d.
Lyapunov Theorem for Nonautonomous Systems

**Theorem:**

- **Stability:** Let $x = 0$ be an Equ. pt. for $\dot{x} = f(t, x)$ and $D \subseteq \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : [0, \infty) \times D \rightarrow \mathbb{R}$ be a continuously differentiable function s.t.:
  1. $V$ is p.d. $\equiv V(x, t) \geq \alpha(\|x\|)$, $\alpha$ is class $\mathcal{K}$
  2. $\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x)$ is n.s.d.

then $x = 0$ is **stable**.

- **Uniform Stability:** If, furthermore

  3. $V$ is decrescent $\equiv V(x, t) \leq \beta(\|x\|)$, $\beta$ is class $\mathcal{K}$

then the origin is **uniformly stable**.

- **Uniform Asymptotic Stability:** If, furthermore conditions 2, 3 is strengthened by

  $$\dot{V} \leq -W_3(x)$$

where $W_3$ is a p.d. fcn. In other word, $\dot{V}$ is n.d., then the origin is **uniformly asymptotically stable**.
Theorem (continued):

- **Global Uniform Asymptotic Stability:** If, the conditions above are satisfied globally $\forall x \in \mathbb{R}^n$, and
  
  4. $V$ is radially unbounded $\equiv \alpha$ is class $\mathcal{K}_\infty$

then the origin is **globally uniformly asymptotically stable**.

- **Exponential Stability:** If, the conditions above are satisfied with $w_i(r) = k_i r^c$, $i = 1, \ldots, 3$ for some positive constants $k_i$ & $c$:

  
  $$\begin{align*}
  k_1\|x\|^c & \leq V(t, x) \leq k_2\|x\|^c \\
  \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) & \leq -k_3\|x\|^c, \quad \forall x \in D,
  \end{align*}$$

then $x = 0$ is **exponentially stable**.

Moreover, if the assumptions hold globally, then the origin is **globally exponentially stable**.
Nonautonomous Systems

- **Example:** Consider \( \dot{x} = -[1 + g(t)] x^3 \)

  where \( g(t) \) is cont. and \( g(t) \geq 0 \) for all \( t \geq 0 \).
  - Let \( V(x) = x^2/2 \), then
    \[
    \dot{V} = -[1 + g(t)] x^4 \leq -x^4, \quad \forall x \in \mathbb{R} \quad \text{and} \quad t \geq 0
    \]
    - All assumptions of the theorem are satisfied with \( W_1(x) = W_2(x) = V(x) \) and \( W_3(x) = x^4 \). Hence, the origin is **g.u.a.s.**

- **Example:** Consider
  \[
  \begin{align*}
  \dot{x}_1 &= -x_1 - g(t)x_2 \\
  \dot{x}_2 &= x_1 - x_2
  \end{align*}
  \]

  where \( g(t) \) is cont. diff, and satisfies
  \( 0 \leq g(t) \leq k \), and \( \dot{g}(t) \leq g(t), \forall t \geq 0 \)
  - Let \( V(t, x) = x_1^2 + [1 + g(t)] x_2^2 \)
Nonautonomous Systems

Example (Cont’d)

\[ x_1^2 + x_2^2 \leq V(t, x) \leq x_1^2 + (1 + k)x_2^2, \quad \forall x \in \mathbb{R}^2 \]

- \( V(t, x) \) is p.d., decrescent, and radially unbounded.

\[ \dot{V} = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2 \]

- We have \( 2 + 2g(t) - \dot{g}(t) \geq 2 + 2g(t) - g(t) \geq 2 \). Then,

\[ \dot{V} \leq -2x_1^2 + 2x_1x_2 - 2x_2^2 = - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleq -x^T Qx \]

where \( Q \) is p.d. \( \implies \dot{V}(t, x) \) is n.d.

- All assumptions of the theorem are satisfied globally with p.d. quadratic fcns \( W_1, W_2, \) and \( W_3 \).

- Recall: for a quadratic fcn \( x^T Px \lambda_{\min}(P)\|x\|^2 \leq x^T Px \leq \lambda_{\max}(P)\|x\|^2 \)

The conditions of exponential stability are satisfied with \( c = 2 \),

\( \therefore \) origin is g.e.s.
Example: Importance of decrescence condition

Consider \( \dot{x}(t) = \frac{\dot{g}(t)}{g(t)}x \)

\( g(t) \) is cont. diff fcn., coincides with \( e^{-t/2} \) except around some peaks where it reaches 1. s.t.:
\[
\int_0^\infty g^2(r)dr < \int_0^\infty e^{-r}dr + \sum_{n=1}^\infty \frac{1}{2^n} = 2
\]

Let \( V(x, t) = \frac{x^2}{g^2(t)}[3 - \int_0^t g^2(r)dr] \rightarrow V \) is p.d (\( V(x, t) > x^2 \))

\( \dot{V} = -x^2 \) is n.d.

But \( x(t) = \frac{g(t)}{g(t_0)}x(t_0) \)

\( \therefore \) origin is not u.a.s.
Linear Time-Varying Systems $\dot{x} = A(t)x$

- The sol. is the so called state transition matrix $\phi(t, t_0)$, i.e., $x(t) = \phi(t, t_0)x(t_0)$

- **Theorem:** The Equ. pt. $x = 0$ is g.u.a.s iff the state transition matrix satisfies $\|\phi(t, t_0)\| \leq ke^{-\gamma(t-t_0)} \forall t \geq t_0 > 0$ for some positive const. $k$&$\gamma$

- **u.a.s.** of $x = 0$ is equivalent to **e.s.** for linear systems.

- Tools/intuitions of TI systems are **no longer valid** for TV systems.

- **Example:**

  \[ \ddot{x} + c(t)\dot{x} + k_0x = 0 \]

  A mass-spring-damper system with t.v. damper $c(t) \geq 0$.

- origin is an Equ. pt. of the system

- Physical intuition may suggest that the origin is **a.s.** as long as the damping $c(t)$ remains strictly positive (implying a constant dissipation of energy) as is for autonomous mass-spring-damper systems.
Linear Time-Varying Systems

▶ Example (cont’d)

▶ HOWEVER, this is not necessarily true:

\[ \ddot{x} + (2 + e^t)x + k_0x = 0 \]

▶ The sol. for \( x(0) = 2, \dot{x}(0) = -1 \) is \( x(t) = 1 + e^{-t} \) which approaches to \( x = 1! \)

▶ Example:

\[
\begin{bmatrix}
-1 + 1.5\cos^2 t & 1 - 1.5\sin t \cos t \\
-1 - 1.5\sin t \cos t & -1 + 1.5\sin^2 t
\end{bmatrix}
\]

▶ For all \( t \), \( \lambda\{A(t)\} = -0.25 \pm j\cdot0.25\sqrt{7} \implies \lambda_1 \& \lambda_2 \) are independent of \( t \) & lie in LHP.

▶ HOWEVER, \( x = 0 \) is unstable

\[
\phi(t, 0) = \begin{bmatrix}
e^{5t} \cos t & e^{-t} \sin t \\
-e^{5t} \sin t & e^{-t} \cos t
\end{bmatrix}
\]
Linear Time-Varying Systems

**Important:** For linear time-varying systems, eigenvalues of $A(t)$ cannot be used as a measure of stability.

**A simple result:** If all eigenvalues of the symmetric matrix $A(t) + A^T(t)$ (all of which are real) remain strictly in LHP, then the LTV system is a.s.:

$$\exists \lambda > 0, \ \forall i, \ \forall t \geq 0, \ \lambda_i\{A(t) + A^T(t)\} \leq -\lambda$$

- Consider the Lyap. fcn candidate $V = x^T x$:

$$\dot{V} = x^T \dot{x} + \dot{x}^T x = x^T (A(t) + A^T(t)) x \leq -\lambda x^T x = -\lambda V$$

  hence, $\forall t \geq 0, \ 0 \leq x^T x = V(t) \leq V(0)e^{-\lambda t}$

- $x$ tends to zero exponentially. Only a **sufficient condition**, though
Linear Time-Varying Systems

**Theorem:** Let \( x = 0 \) be a e.s. Equ. pt. of \( \dot{x} = A(t)x \). Suppose, \( A(t) \) is cont. & bounded. Let \( Q(t) \) be a cont., bounded, p.d. and symmetric matrix, i.e \( 0 < c_3 I \leq Q(t) \leq c_4 I, \ \forall \ t \geq 0 \). Then, there exists a cont. diff., bounded, symmetric, p.d. matrix \( P(t) \) satisfying

\[
\dot{P}(t) = -(A^T(t)P(t) + P(t)A(t) + Q(t))
\]

Hence, \( V(t, x) = x^T P(t)x \) is a Lyap. fcn for the system satisfying e.s. theorem

- \( P(t) \) is symmetric, bounded, p.d. matrix, i.e.
  \[ 0 < c_1 I \leq P(t) \leq c_2 I, \ \forall \ t \geq 0 \]
- \( \therefore c_1 \|x\|^2 \leq V(t, x) \leq c_2 \|x\|^2 \)
- \( \dot{V}(t, x) = x^T \left[ \dot{P}(t) + P(t)A(t) + A^T(t)P(t) \right] x = -x^T Q(t)x \leq - c_3 \|x\|^2 \)
- The conditions of exponential stability are satisfied with \( c = 2 \), the origin is g.e.s.
Linear Time-Varying Systems

- As a special case when $A(t) = A$, then $\phi(\tau, t) = e^{(\tau-t)A}$ which satisfies $\|\phi(t, t_0)\| \leq e^{-\gamma(t-t_0)}$ when $A$ is a stable matrix.

- Choosing $Q = Q^T > 0$, then $P(t)$ is given by

$$P = \int_{t}^{\infty} e^{(\tau-t)A^T} Q e^{(\tau-t)A} d\tau = \int_{0}^{\infty} e^{A^Ts} Q e^{As} ds$$

independent of $t$ and is a solution to the Lyap. equation.
Consider $\dot{x} = f(t, x)$ where $f : [0, \infty) \times D \rightarrow \mathbb{R}^n$ is cont. diff. and $D = \{x \in \mathbb{R}^n \mid \|x\| < r\}$. Let $x = 0$ be an Equ. pt. Also, let the Jacobian matrix be bounded and Lip. on $D$ uniformly in $t$, i.e.

$$\left\| \frac{\partial f}{\partial x}(t, x) \right\| \leq k \quad \forall x \in D, \quad \forall t \geq 0$$

$$\left\| \frac{\partial f}{\partial x}(t, x_1) - \frac{\partial f}{\partial x}(t, x_2) \right\| \leq L\|x_1 - x_2\| \quad \forall x_1, x_2 \in D, \quad \forall t \geq 0$$

Let $A(t) = \left. \frac{\partial f}{\partial x}(t, x) \right|_{x=0}$. Then, $x = 0$ is e.s. for the nonlinear system if it is an e.s. Equ. pt. for the linear system $\dot{x} = A(t)x$. 
Now, exponential stability of the linearization is a necessary and sufficient condition for e.s. of \( x = 0 \)

**Theorem:** Let \( x = 0 \) be an Equ. pt. of \( \dot{x} = f(t, x) \) with conditions as above. Let \( A(t) = \frac{\partial f(t, x)}{\partial x} \bigg|_{x=0} \). Then, \( x = 0 \) is an e.s. Equ. pt. for the nonlinear system iff it is an e.s. Equ. pt. for the linear system \( \dot{x} = A(t)x \). For autonomous systems e.s. condition is satisfied iff \( A \) is Hurwitz.

**Example:**

\[ \dot{x} = -x^3 \]

Recall that \( x = 0 \) is a.s.

However, linearization results in \( \dot{x} = 0 \) whose \( A \) is not Hurwitz.

Using the above theorem, we conclude that \( x = 0 \) is **not exponentially stable** for nonlinear system.
Summary

- **Lyapunov Theorem for Nonautonomous Systems** $\dot{x} = f(x, t)$:
  - **Stability**: Let $x = 0$ be an Equ. pt. and $D \in \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : [0, \infty) \times D \rightarrow \mathbb{R}$ be a continuously differentiable function s.t.: $V$ is p.d., and $\dot{V}$ is n.s.d.
  - **Uniform Stability**: If, furthermore $V$ is decrescent
  - **Uniform Asymptotic Stability**: If, furthermore $\dot{V}$ is n.d.
  - **Global Uniform Asymptotic Stability**: If, the conditions above are satisfied globally $\forall x \in \mathbb{R}^n$, and $V$ is radially unbounded
  - **Exponential Stability**: If for some positive constants $k_i$ & $c$:
    
    $$ k_1 \|x\|^c \leq V(t, x) \leq k_2 \|x\|^c $$
    
    $$ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -k_3 \|x\|^c, \forall x \in D, $$
    
    then $x = 0$ is **exponentially stable**
  - Moreover, if the assumptions hold globally, then the origin is **globally exponentially stable**
Summary

- **Linear Time-Varying Systems** \( \dot{x} = A(t)x \)
  - **Theorem:** The Equ. pt. \( x = 0 \) is **g.u.a.s** iff the state transition matrix satisfies \( \|\phi(t, t_0)\| \leq ke^{-\gamma(t-t_0)} \forall t \geq t_0 > 0 \) for some positive const. \( k \& \gamma \)
  - If all eigenvalues of the symmetric matrix \( A(t) + A^T(t) \) remain strictly in LHP, then the LTV system is **a.s**
  - For LTV systems, eigenvalues of \( A(t) \) alone cannot be used as a measure of stability.
  - **Theorem:** Let \( x = 0 \) be a **e.s.** Equ. pt. of \( \dot{x} = A(t)x \). Suppose, \( A(t) \) is cont. & bounded. Let \( Q(t) \) be a cont., bounded, p.d. and symmetric matrix, i.e \( 0 < c_3 I \leq Q(t) \leq c_4 I, \forall t \geq 0 \). Then, there exists a cont. diff., bounded, symmetric, p.d. matrix \( P(t) \) satisfying

\[
\dot{P}(t) = -(A^T(t)P(t) + P(t)A(t) + Q(t))
\]

Hence, \( V(t, x) = x^TP(t)x \) is a Lyap. fcn for the system satisfying e.s. theorem
Barbalat’s Lemma

- For autonomous systems, invariant set theorems are power tools to study asymptotic stability when $\dot{V}$ is n.s.d.
- The invariant set theorem is not valid for nonautonomous systems.
- Hence, asymptotic stability of nonautonomous systems is generally more difficult than that of autonomous systems.
- An important result that remedy the situation: Barbalat’s Lemma

Asymptotic Properties of Functions and Their Derivatives:

- For diff. fcn $f$ of time $t$, always keep in mind the following three facts!
  1. $\dot{f} \longrightarrow 0 \not\Rightarrow f$ converges
     - The fact that $\dot{f} \longrightarrow 0$ does not imply $f(t)$ has a limit as $t \longrightarrow \infty$.
     - **Example:** $f(t) = \sin(ln t) \sim \dot{f} = \frac{\cos(ln t)}{t} \longrightarrow 0$ as $t \longrightarrow \infty$
     - However, the fcn $f(t)$ keeps oscillating (slower and slower).
     - **Example:** For an unbounded function $f(t) = \sqrt{t} \sin(ln t)$, $\sim \dot{f} = \frac{\sin(ln t)}{2\sqrt{t}} + \frac{\cos(ln t)}{\sqrt{t}} \longrightarrow 0$ as $t \longrightarrow \infty$
Asymptotic Properties of Functions and Their Derivatives:

2. \( f \) converges \( \nRightarrow \dot{f} \rightarrow 0 \)
   - The fact that \( f(t) \) has a finite limit at \( t \rightarrow \infty \) does not imply that \( \dot{f} \rightarrow 0 \).
   - **Example:** \( f(t) = e^{-t} \sin(e^{2t}) \rightarrow 0 \) as \( t \rightarrow \infty \)
   - while its derivative \( \dot{f} = -e^{-t} \sin(e^{2t}) + 2e^{t} \cos(e^{2t}) \) is unbounded.

3. If \( f \) is lower bounded and decreasing (\( \dot{f} \leq 0 \)), then it converges to a limit.
   - However, it does not say whether the slope of the curve will diminish or not.

Given that a fcn tends towards a finite limit, what additional property guarantees that the derivatives converges to zero?
Barbalat’s Lemma

Barbalat’s Lemma: If the differentiable fcn has a finite limit as $t \to \infty$, and if $\dot{f}$ is uniformly cont., then $\dot{f} \to 0$ as $t \to \infty$.

- proved by contradiction

A function $g(t)$ is **continuous** on $[0, \infty)$ if

$$\forall t_1 \geq 0, \forall R > 0, \exists \eta(R, t_1) > 0, \forall t \geq 0, \quad |t - t_1| < \eta \implies |g(t) - g(t_1)| < R$$

A function $g(t)$ is **uniformly continuous** on $[0, \infty)$ if

$$\forall R > 0, \exists \eta(R) > 0, \forall t_1 \geq 0, \forall t \geq 0, \quad |t - t_1| < \eta \implies |g(t) - g(t_1)| < R$$

i.e. an $\eta$ can be found independent of specific point $t_1$. 
Barbalat’s Lemma

▶ A **sufficient condition** for a diff. fcn. to be **uniformly continuous** is that its **derivative** be **bounded**
  
  ▶ This can be seen from Mean-Value Theorem:

  \[ \forall t, \forall t_1, \exists t_2 \text{ (between } t \text{ and } t_1 \text{) s.t. } g(t) - g(t_1) = \dot{g}(t_2)(t - t_1) \]

  ▶ if \( \dot{g} \leq R_1 \forall t \geq 0 \), let \( \eta = \frac{R}{R_1} \) independent of \( t_1 \) to verify the definition above.

▶ ∴ An immediate and practical corollary of Barbalat’s lemma: If the differentiable fcn \( f(t) \) has a finite limit as \( t \to \infty \) and \( \ddot{f} \) exists and is bounded, then \( \dot{f} \to 0 \) as \( t \to \infty \)
Using Barbalat’s lemma for Stability Analysis

**Lyapunov-Like Lemma:** If a scaler function $V(t, x)$ satisfies the following conditions

1. $V(t, x)$ is lower bounded
2. $\dot{V}(t, x)$ is negative semi-definite
3. $\dot{V}(t, x)$ is uniformly continuous in time

then $\dot{V}(t, x) \to 0$ as $t \to \infty$.

**Example:**

Consider a simple adaptive control systems:

$$\dot{e} = -e + \theta w(t)$$
$$\dot{\theta} = -ew(t)$$

where $e$ is the tracking error, $\theta$ is the parameter error, and $w(t)$ is a bounded continuous fcn.
Example (Cont’d)

Consider the lower bounded fcn:

\[ V = e^2 + \theta^2 \]

\[ \dot{V} = 2e(-e + \theta w) + 2\theta(-ew(t)) = -2e^2 \leq 0 \]

\[ \therefore V(t) \geq V(0), \text{ therefore, } e \text{ and } \theta \text{ are bounded.} \]

Invariant set theorem cannot be used to conclude the convergence of \( e \), since the dynamic is nonautonomous.

To use Barbalat’s lemma, check the uniform continuity of \( \dot{V} \).

\[ \ddot{V} = -4e(-e + \theta w) \]

\( \ddot{V} \) is bounded, since \( w \) is bounded by assumption and \( e \) and \( \theta \) are shown to be bounded \( \leadsto \) \( \dot{V} \) is uniformly continuous

Applying Barbalat’s lemma: \( \dot{V} = 0 \implies e \to 0 \text{ as } t \to \infty. \)

Important: Although \( e \to 0 \), the system is not a.s. since \( \theta \) is only shown to be bounded.
Boundedness and Ultimate Boundedness

- Lyapunov analysis can be used to show boundedness of the solution even when there is no Equ. pt.

- **Example:**

\[
\dot{x} = -x + \delta \sin t, \quad x(t_0) = a, \quad a > \delta > 0
\]

which has no Equ. pt and

\[
x(t) = e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin \tau \, d\tau
\]

\[
x(t) \leq e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \, d\tau
\]

\[
= e^{-(t-t_0)}a + \delta \left[ 1 - e^{-(t-t_0)} \right] \leq a, \quad \forall \ t \geq t_0
\]

- The solution is bounded for all \( t \geq t_0 \), uniformly in \( t_0 \).
- The bound is conservative due to the exponentially decaying terms.
Example Cont’d

- Pick any number $b$ s.t. $\delta < b < a$, it can be seen that
  
  $$|x(t)| \leq b, \quad \delta \forall t \geq t_0 + \ln \left( \frac{a - \delta}{b - \delta} \right)$$

- $b$ is also independent of $t_0$
  - The solution is said to be **uniformly ultimately bounded**
  - $b$ is called the **ultimate bound**

- The same properties can be obtained via Lyap. analysis. Let $V = x^2/2$, then
  
  $$\dot{V} = xx = -x^2 + x\delta \sin t \leq -x^2 + \delta|x|$$

- $\dot{V}$ is not n.d., bc. near the origin, positive linear term $\delta|x|$ is dominant.

- However, $\dot{V}$ is negative outside the set $\{|x| \leq \delta\}$.

- Choose, $c > \delta^2/2$, solutions starting in the set $\{V(x) < c\}$ will remain there in for all future time since $\dot{V}$ is negative on the boundary $V = c$.

- Hence, the solution is **ultimately bounded**.
**Boundedness and Ultimate Boundedness**

- **Definition:** The solutions of $\dot{x} = f(t, x)$ where $f : (0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in $t$ and locally Lipschitz in $x$ on $(0, \infty) \times D$, and $D \subseteq \mathbb{R}^n$ is a domain that contains the origin are

  - **uniformly bounded** if there exist a positive constant $c$, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, independent of $t_0$, s.t.
    \[
    \|x(t_0)\| \leq a \implies \|x(t)\| \leq \beta, \quad \forall \ t \geq t_0
    \] (5)

  - **uniformly ultimately bounded** if there exist positive constants $b$ and $c$, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $T = T(a, b) > 0$, independent of $t_0$, s.t.
    \[
    \|x(t_0)\| \leq a \implies \|x(t)\| \leq b, \quad \forall \ t \geq t_0 + T
    \] (6)

- **globally uniformly bounded** if (5) holds for arbitrary large $a$.
- **globally uniformly ultimately bounded** if (6) holds for arbitrary large $a$. 
How to Find Ultimate Bound?

▶ In many problems, negative definiteness of $\dot{V}$ is guaranteed by using norm inequalities, i.e.:

$$\dot{V}(t, x) \leq -W_3(x), \quad \forall \mu \leq \|x\| \leq r, \quad \forall t \geq t_0 \quad (7)$$

▶ If $r$ is sufficiently larger than $\mu$, then $c$ and $\epsilon$ can be found s.t. the set $\Lambda = \{\epsilon \leq V \leq c\}$ is nonempty and contained in $\{\mu \leq \|x\| \leq r\}$.

▶ If the Lyap. fcn satisfy:

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$$

for some class $\mathcal{K}$ functions $\alpha_1$ and $\alpha_2$.

▶ We are looking for a bound on $\|x\|$ based on $\alpha_1$ and $\alpha_2$ s.t. satisfies (7).
How to Find Ultimate Bound?

- We have: $V(x) \leq c \implies \alpha_1(\|x\|) \leq c \iff \|x\| \leq \alpha_1^{-1}(c)$
  
  $\therefore c = \alpha_1(r)$ ensures that $\Omega_c \subset B_r$.

- On the other hand we have:
  
  $\|x\| \leq \mu \implies V(x) \leq \alpha_2(\mu)$

- Hence, taking $\epsilon = \alpha_2(\mu)$ ensures that $B_\mu \subset \Omega_\epsilon$.

- To obtain $\epsilon < c$, we must have $\mu < \alpha_2^{-1}(\alpha_1(r))$.

- Hence, all trajectories starting in $\Omega_c$ enter $\Omega_\epsilon$ within a finite time $T$ as discussed before.

- To obtain the ultimate bound on $x(t)$,

  $V(x) \leq \epsilon \implies \alpha_1(\|x\|) \leq \epsilon \iff \|x\| \leq \alpha_1^{-1}(\epsilon)$

- Recall that $\epsilon = \alpha_2(\mu)$, hence: $x \in \Omega_\epsilon \implies \|x\| \leq \alpha_1^{-1}(\alpha_2(\mu))$

  $\therefore$ The ultimate bound can be taken as $b = \alpha_1^{-1}(\alpha_2(\mu))$. 
Ultimate Boundedness

**Theorem:** Let $D \in \mathbb{R}^n$ be a domain containing the origin and $V : [0, \infty) \times D \rightarrow \mathbb{R}$ be a cont. diff. fcn s.t.

\[
\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \tag{8}
\]

\[
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \quad \forall \|x\| \geq \mu > 0 \tag{9}
\]

\forall t \geq 0 and \forall x in D, where $\alpha_1$ and $\alpha_2$ are class $\mathcal{K}$ fcns and $W_3(x)$ is a cont. p.d. fcn. Take $r > 0$ s.t. $B_r \subset D$ and suppose that $\mu < \alpha_2^{-1}(\alpha_1(r))$

Then, there exists a class $\mathcal{KL}$ fcn $\beta$ and for every initial state $x(t_0)$ satisfying $\|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$, there is $T \geq 0$ (dependent on $x(t_0)$ and $\mu$ s.t.)

\[
\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall \ t_0 \leq t \leq t_0 + T
\]

\[
\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\mu)) \quad \forall \ t \geq t_0 + T
\]

Moreover, if $D = \mathbb{R}^n$ and $\alpha_1 \in \mathcal{K}_\infty$, then the above inequalities hold for any initial state $x(t_0)$. 
Ultimate Boundedness

- The inequalities of the theorem show that
  - $x(t)$ is uniformly bounded for all $t \geq t_0$
  - uniformly ultimately bounded with the ultimate bound $\alpha_1^{-1}(\alpha_2(\mu))$.
  - The ultimate function is a class $\mathcal{K}$ fcn of $\mu \mapsto$ the smaller the value of $\mu$, the smaller the ultimate bound
  - As $\mu \rightarrow 0$, the ultimate bound approaches zero.

- The main application of this theorem arises in studying the stability of perturbed system.
Example for Ultimate Boundedness

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -(1 + x_1^2)x_1 - x_2 + M \cos \omega t \]

where \( M > 0 \)

- Let \( V(x) = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \frac{1}{2} x_1^4 = x^T P x + \frac{1}{2} x_1^4 \)

- \( V(x) \) is p.d. and radially unbounded \( \Rightarrow \) there exist class \( \mathcal{K}_\infty \) fcns. \( \alpha_1 \) and \( \alpha_2 \) satisfying (8).

- \( \dot{V} = -x_1^2 - x_1^4 - x_2^2 + (x_1 + 2x_2)M \cos \omega t \leq -\|x\|^2_2 - x_1^4 + M\sqrt{5}\|x\|_2 \)

  where \( (x_1 + 2x_2) = [1 \ 2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \sqrt{5}\|x\|_2 \)

- We want to use part of \( -\|x\|^2_2 \) to dominate \( M\sqrt{5}\|x\|_2 \) for large \( \|x\| \)

- \( \dot{V} \leq -(1 - \theta)\|x\|^2_2 - x_1^4 - \theta\|x\|^2_2 + M\sqrt{5}\|x\|_2 \), for \( 0 < \theta < 1 \)

- Then \( \dot{V} \leq -(1 - \theta)\|x\|^2_2 - x_1^4 \quad \forall \|x\|_2 \geq \frac{M\sqrt{5}}{\theta} \Rightarrow \mu = M\sqrt{5}/\theta \)
Example for Ultimate Boundedness

- the solutions are u.u.b.
- Next step: finding ultimate bound:

\[ V(x) \geq x^T P x \geq \lambda_{min}(P) \|x\|_2^2 \]
\[ V(x) \leq x^T P x + \frac{1}{2} \|x\|_2^4 \leq \lambda_{max}(P) \|x\|_2^2 + \frac{1}{2} \|x\|_2^4 \]

- \( \alpha_1(r) = \lambda_{min}(P)r^2 \) and \( \alpha_2(r) = \lambda_{max}(P)r^2 + \frac{1}{2}r^4 \)
- \( \therefore \) ultimate bound: \( b = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\lambda_{max}(P)\mu^2 + \mu^4/2\lambda_{min}(P)} \)