Signals and Systems
Lecture 3: Fourier Series(CT)

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Introduction

- Perviously, we have seen that by using convolution, LTI systems can be represented by linear combination of linear impulses.
- Now we are going to describe LTI systems based on linear combination of a set of basic signals (complex exponentials).
- Based on superposition property of LTI systems, response to any input including linear combination of basic signals is the same linear combination of the individual responses to each of the basic signals
- At first, CT/DT periodic signals are described by Fourier Series
- Then Fourier transform is introduced to represent aperiodic signals

LTI Systems Response to Complex Exponential Signals

- For analyzing LTI system, the signals can be represented as a linear combination of basic signals.
- The basic signals should
  1. be able to be used to construct a wide range of useful class of signals
  2. have simple structure in LTI system response.
- Complex exponential signals are good candidate for basic signal since
  - its LTI system response is the same complex exponential with different amplitude
  - for CT: $e^{st} \rightarrow H(s)e^{st}$ ($H(s)$: a function of s)
  - for DT: $z^n \rightarrow H(z)z^n$ ($H(z)$: a function of z)

- **Eigenfunction:** a signal for which the system output is a modulus of input.
  - The constant value which can be complex is called eigenvalue.
- Consider input signal: $x(t) = e^{st}$:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)e^{s(-\tau)}d\tau}_{H(s)} = H(s)e^{st}$$

- $\therefore\ x(t) = e^{st}$ is an eigenfunction
- Consider input signal: $x[n] = z^n$:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k]z^{-k}}_{H(z)} = H(z)z^n$$

- $\therefore\ x[n] = z^n$ is an eigenfunction

- $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t}$
- $x[n] = \sum_{k=-\infty}^{\infty} a_k z_k^n \rightarrow y[n] = \sum_{k=-\infty}^{\infty} a_k H(z_k) z^n$
- For now assume that s and z are purely imaginary:
  - $s = j\omega \rightsquigarrow e^{st} = e^{j\omega t}$
  - $z = e^{j\omega} \rightsquigarrow z^n = e^{i\omega n}$

Fourier Series for CT Signals

- Fourier series can represent CT periodic signals
- Remember the definition of periodic signals: $x(t) = x(t + T)$ with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$
- $x(t) = e^{j\omega t}$ is a periodic signal with fundamental freq. ω_0
- Remember the harmonically related complex exponentials: $\phi_k(t) = e^{jk\omega_0 t} = e^{jk(\frac{2\pi}{T})t}, k = 0, \pm 1, \pm 2, \ldots$
  - Each harmonic has fundamental freq. which is a multiply of ω_0
- A linear combination of harmonically related complex exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi/Tt} \tag{1}$$

  - it is periodic with period T
  - $k = \pm 1$ have fundamental freq. ω_0 (first harmonic)
  - $k = \pm N$ have fundamental freq. $N\omega_0$ (Nth harmonic)

- Equation (1) is Fourier Series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi/Tt}$$

- Now if $x(t)$ is real:
  - $x(t) = x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$
  - $k = -k \rightsquigarrow x(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$
  - $a_k = a_{-k}^*$
  - If a_k is real: $a_k = a_{-k}$.
- $x(t) = a_0 + \sum_{k=1}^{\infty} a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} =$ $a_0 + \sum_{k=1}^{\infty} a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} = a_0 + 2\sum_{k=1}^{\infty} \mathcal{R}e\{a_k e^{jk\omega_0 t}\}$
- Describe a_k by polar representation: $a_k = A_k e^{j\theta_k}$:
  - Second definition of Fourier Series: $x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k cos(k\omega_0 t + \theta_k)$
- Describe a_k by Cartesian representation: $a_k = B_k + jC_k$:
  - Third definition of Fourier Series:
    $x(t) = a_0 + 2\sum_{k=1}^{\infty} B_k cos(k\omega_0 t) - C_k sin(k\omega_0 t)$

How to find coefficient a_k

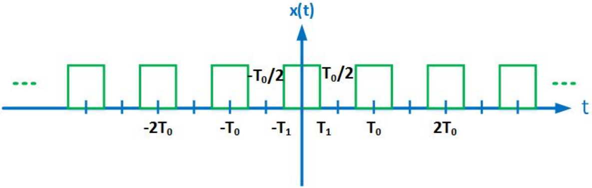
- Rewrite Fourier series: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
- Multiply both sides by $e^{-jn\omega_0 t}$: $x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0}$
- Take $\int_0^T$: $\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}dt$
- T is fundamental period of $x(t)$:

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt \tag{2}$$

- Use Euler's formula:
  $\int_0^T e^{-j(k-n)\omega_0 t}dt = \int_0^T cos((k-n)\omega_0 t)dt + j\int_0^T sin((k-n)\omega_0 t)dt$
- $cos((k-n)\omega_0 t)$ and $sin((k-n)\omega_0 t)$ are periodic with fundamental period: $\frac{T}{|k-n|}$

- $\therefore e^{j(k-n)\omega_0 t} = \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$
- Substituting the above equation to (2): $\int_0^T x(t)e^{-jn\omega_0 t}dt = a_n T$
- Analysis Equation of Fourier Series: $a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$
- Synthesis Equation of Fourier Series: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
- dc or constant component of $x(t)$: $a_0 = \frac{1}{T}\int_T x(t)dt$

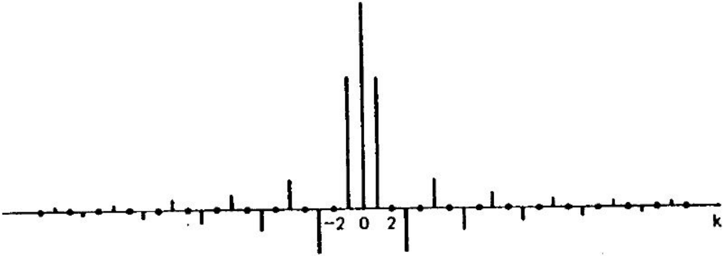
Example: Fourier Series of periodic square wave



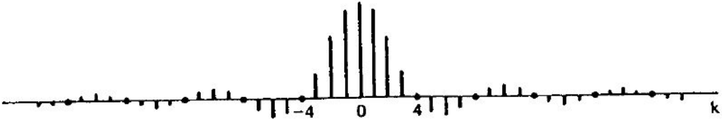
- $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 \leq |t| \leq T/2 \end{cases}$
- dc gain: $a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{2T_1}{T}$
- $k \neq 0$: $a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt =$
  $\frac{-1}{jk\omega_0 T}e^{-jk\omega_0 t}|_{-T_1}^{T_1} = \frac{sink\omega_0 T_1}{k\pi}$

Example Cont'd

- a) $T = 4T_1$, b) $T = 8T_1$



(a)



(b)

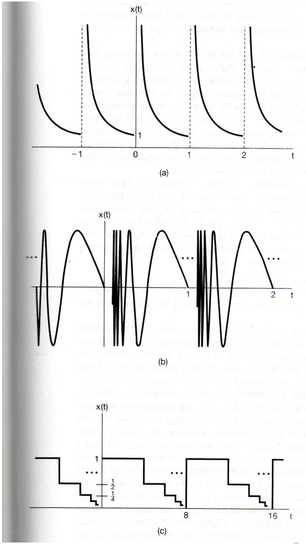
Fourier Series Convergence

- How can one guarantee that $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ represent a periodic signal $x(t)$?
- Let us use an engineering tool named "approximation error":
  - Assume that a periodic signal can be expressed by linear combination of limited complex exponential terms: $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$
- Approximation error e_N is defined: $e_N = x(t) - x_N(t)$
- The energy of the error in one period: $E_N = \int_T |e_N(t)|^2 dt$
- It can be shown that to achieve min E_N, one should define (show it!): $a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$
- $N \uparrow \rightsquigarrow E_N \downarrow$
  - $N \to \infty \rightsquigarrow \lim E_N \to 0$

Fourier Series Convergence

- If $a_k \to \infty$ the approximation will diverge
- Even for bounded a_k the approximation may not be applicable for all periodic signals.
- Convergence Conditions of Fourier Series Approximation
  1. Energy of signal should be a finite in a period: $\int_T |x(t)|^2 dt < \infty$
     - This condition only guarantees $E_N \to 0$
     - It does not guarantee that $x(t)$ equals to its Fourier series at each moment t
  2. Drichlet Conditions
     - Over any period, $x(t)$ must be absolutely integrable: $\int_T |x(t)| dt < \infty$
       - with square integrability condition, it guarantees $|a_k| < \infty$
     - In a single period, $x(t)$ should have finite number of max and min
     - In any finite interval of time, there are only a finite number of discontinuities. Each discontinuity should be finite.

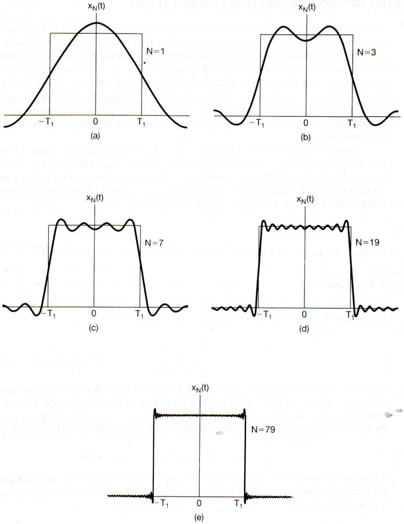
Drichlet Conditions



Gibbs Phenomenon

- It happens for FS of piecewise continuous periodic signals with bounded discontinuities.
- FS produces ripples in the vicinity of discontinuity.
- By Increasing N
  - the ripples concentrates around discontinuities.
  - Total energy of the ripples decreases
  - BUT the amplitude of the largest ripple does not change

Gibb's Phenomenon



Properties of CT Fourier Series

- Consider $x(t)$ and $y(t)$: periodic signals with same fundamental period T
- $x(t) \Leftrightarrow a_k, y(t) \Leftrightarrow b_k$
- Linearity: $Ax(t) + By(t) \Leftrightarrow Aa_k + Bb_k$
- Time Shifting: $x(t - t_0) \Leftrightarrow a_k e^{-jk\omega_0 t_0}$
  - In time shifting magnitude of $\mathcal{F}s$ coefficient remains the same but its angel is changed
- Time Reversal: $x(-t) \Leftrightarrow a_{-k}$
  - even $x(t) \Leftrightarrow$ even a_k: $a_k = a_{-k}$
  - odd $x(t) \Leftrightarrow$ odd a_k: $a_k = -a_{-k}$
- Time Scaling $x(\alpha t) \Leftrightarrow a_k$ but fundamental period $\frac{T}{\alpha}$

Properties of CT Fourier Series

- Multiplication: $x(t)y(t) \Leftrightarrow h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$ (DT convolution between coefficients)
- Conjugation and Conjugate Symmetry:
  - Real $x(t) \Leftrightarrow a_{-k} = a_k^*$ (conjugate symmetric)
  - Real and even $x(t) \Leftrightarrow a_k = a_k^*$ (real and even a_k)
  - Real and odd $x(t) \Leftrightarrow a_k = -a_k^*$ (purely imaginary and odd a_k), $a_0 = 0$
  - Even part of $x(t) \Leftrightarrow \mathcal{R}e\{a_k\}$ (show it!)
  - Odd part of $x(t) \Leftrightarrow j\mathcal{I}m\{a_k\}$ (show it!)
- Periodic convolution: $x(t) * y(t)$ (for one period) $= \int_{-T/2}^{T/2} x(\tau)y(t-\tau)d\tau \Leftrightarrow Ta_k b_k$
- Parseval's Relation: $\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$
  - Total average power = sum of average power in all harmonic components
  - Energy in time domain equals to energy in frequency domain