Nonlinear Control
Lecture 4: Stability Analysis I

Farzaneh Abdollahi

Department of Electrical Engineering

Amirkabir University of Technology

Fall 2010

Stability

- Stability theory is divided into three parts:
  1. Stability of equilibrium points
  2. Stability of periodic orbits
  3. Input/output stability
- An equilibrium point (Equ. pt.) is:
  - Stable if all solutions starting at nearby points stay nearby.
  - Asymptotically Stable if all solutions starting at nearby points not only stay nearby, but also tend to the Equ. pt. as time approaches infinity.
  - Exponentially Stable, if the rate of converging to the Equ. pt. is exponentially.
- Lyapunov stability theorems give sufficient conditions for stability, asymptotic stability, and so on.
- Lyapunov stability analysis can be used to show boundedness of the solution even when the system has no equilibrium points.
- The theorems provide necessary conditions for stability are so-called converse theorems.

- The most popular method for studying stability of nonlinear systems is introduced by a Russian mathematician named Alexander Mikhailovich Lyapunov
- Lyapunov's work " *The General Problem of Motion Stability* published in 1892 includes two methods:
  - Linearization Method: studies nonlinear local stability around an Equ. point from stability properties of its linear approximation
  - Direct Method: not restricted to local motion. Stability of nonlinear system is studied by proposing a scalar *energy-like* function for the system and examining its time variation
- His work was then introduced by other scientists like Poincare and Lasalle

http://en.wikipedia.org/wiki/Aleksandr_Lyapunov

Autonomous Systems

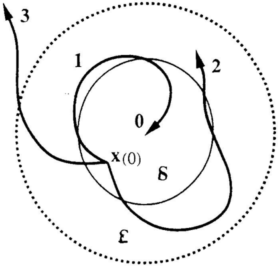
- Consider the autonomous system:
$$\dot{x} = f(x)$$
where $f : D \longrightarrow R^n$ is a locally Lip. function on a domain $D \subset R^n$.
- Let $\bar{x} \in D$ be an Equ. point, that is $f(\bar{x}) = 0$.
- **Objective**: To characterize stability of $\bar{x}$.
- without loss of generality (*wlog*), let $\bar{x} = 0$
  - If $\bar{x} \neq 0$, introduce a coordinate transformation: $y = x - \bar{x}$, then
  - $\dot{y} = \dot{x} = f(y + \bar{x}) = g(y)$ with $g(0) = 0$

- The Equ. point $x = 0$ of $\dot{x} = f(x)$ is:
  - **stable**, if for each $\epsilon > 0$, $\exists\ \delta = \delta(\epsilon) > 0$ s.t.

$$\|x(0)\| < \delta \Longrightarrow \|x(t)\| < \epsilon \quad \forall t \geq 0$$

  - **unstable**, if it is not stable
  - **asymptotically stable**, if it is stable and δ can be chosen s.t.

$$\|x(0)\| < \delta \Longrightarrow \lim_{t \longrightarrow \infty} x(t) = 0$$



curve 1 - asymptotically stable

curve 2 - marginally stable

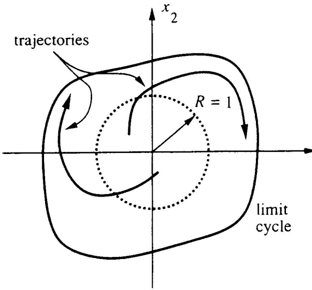
curve 3 - unstable

- $\therefore$ Lyapunov stability means that the system trajectory can be kept arbitrary close to the origin by starting sufficiently close to it.
- An Equ. point which is Lypunov stabile but not asymptotically stable is called Marginally stable

- Example: Van Der Pol Oscillator
  - Van der pol oscillator dynamics:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + (1-x_1)^2 x_2\end{aligned}$$

  - All system trajectories start except from origin, asymptotically approaches a limit cycle.
  - $\therefore$ Even the system states remain around the Equ. point in a certain sense, they can not stay arbitrarily close to it.
  - So the Equ. point is unstable.
- Implicit in Lyapunov stability condition is that the sol. are defined $\forall t \geq 0$.
- This is not guaranteed by local Lip.
- The additional condition imposed by Lyapunov theorem will ensure global existence of sol.



Unstable origin of the Van der Pol

Lyapunov Stability

- **Physical Motivation**
- Consider the pendulum example (recall Lecture 2):

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-g}{l} sin x_1 - \frac{k}{m} x_2\end{aligned}$$

- In first period it has two Equ. pts. $(x_1 = 0, x_2 = 0)$ & $(x_1 = \pi, x_2 = 0)$
- For frictionless pendulum, i.e. $k = 0$: trajectories are closed orbits in neighborhood of 1^{st} Equ. pt. $\rightsquigarrow \epsilon - \delta$ requirement for stability is satisfied.
- However, it is not asymptotically stable.
- For Pendulum with friction, i.e. $k > 0$
  the 1^{st} Equ. pt. is a stable focus $\rightsquigarrow \epsilon - \delta$ requirement for asymptotic stability is satisfied.
  the 2^{nd} Equ. pt. is a saddle point $\rightsquigarrow \epsilon - \delta$ requirement is not satisfied $\rightsquigarrow$ it is unstable

- To generalize the phase-plane analysis, consider the energy associated with the pendulum:

$$E(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} \frac{g}{l} sin\ y\ dy = \frac{1}{2}x_2^2 + \frac{g}{l}(1 - cosx_1), \quad E(0) = 0$$

- If $k = 0$, system is conservative, i.e. there is no dissipation of energy:
  - $E =$ constant during the motion of the system.
  - $\therefore \frac{dE}{dt} = 0$ along the traj. of the system.
- If $k > 0$, energy is being dissipated
  - $\frac{dE}{dt} < 0$ along the traj. of the system.
  - $\therefore E$ starts to decrease until it eventually reaches zero, at that pt. $x = 0$.
- Lyapunov showed that certain other function can be used instead of energy function to determine stability of an Equ. pt.

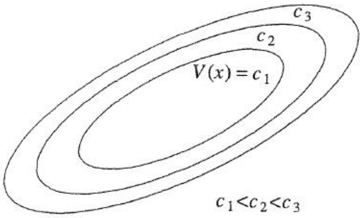
Lyapunov's Direct Method:

- *Let $x = 0$ be an Equ. pt. for $\dot{x} = f(x)$. Let $V : D \longrightarrow R,\ D \subset R^n$ be a continuously differentiable function on a neighborhood D of $x = 0$, s.t.*
  1. $V(0) = 0$
  2. $V(x) > 0$ in $D - \{0\}$
  3. $\dot{V}(x) \leq 0$ in D

  Then $x = 0$ is **stable**.

  Moreover, if $\dot{V}(x) < 0$ in $D - \{0\}$ then $x = 0$ is **asymptotically stable**.
- The continuously differentiable function $V(x)$ is called a **Lyapunov function**.
- The surface $V(x) = c$, for some $c > 0$ is called a **Lyapunov surface** or **level surface**.

Lyapunov Stability



Level surfaces of a Lyapunov function.

- when $\dot{V} \leq 0 \rightsquigarrow$
  when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\Omega_c = \{x \in R^n | V(x) \leq c\}$ and traps inside Ω_c.
- when $\dot{V} < 0 \rightsquigarrow$
  trajectories move from one level surface to an inner level with smaller c till $V(x) = c$ shrinks to zero as time goes on

Lyapunov Stability

- A function satisfying $V(0) = 0$ & $V(x) > 0$ in $D - \{0\}$ is said to be Positive Definite (p.d.)
- If it satisfies a weaker condition $V(x) \geq 0$ for $x \neq 0$ is said to be Positive Semi-Definite (p.s.d.)
- A function is Negative Definite (n.d.) or Negative Semi-Definite (n.s.d.) if $-V(x)$ is p.d. or p.s.d., respectively.
- Lyapunov theorem states that:
  The origin is stable if there is a continuously differentiable, p.d. function $V(x)$ s.t. $\dot{V}(x)$ is n.s.d., and is asymptotically if $\dot{V}(x)$ is n.d.
- Note that when x is a vector:

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

Lyapunov Stability

- A class of scalar functions for which sign definition can be easily checked is "**quadratic functions:**"

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j P_{ij}$$

where $P = P^T$ is a real matrix.

- $V(x)$ is p.d./p.s.d. iff $\lambda_i\{P\} > 0$ or $\lambda_i\{P\} \geq 0, \ i = 1 \ldots n$
- $\lambda_i\{P\} > 0$ or $\lambda_i\{P\} \geq 0, \ i = 1 \ldots n$ iff all leading principle minors of P are positive or non-negative, respectively.
- If $V(x)$ is p.d. (p.s.d.), we say the matrix P is p.d. (p.s.d.) and write $P > 0$ ($P \geq 0$).

Example 1

$$
\begin{aligned}
V(x) &= ax_1^2 + 2x_1x_3 + ax_2^2 + 4x_2x_3 + ax_3^2 \\
&= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T P x
\end{aligned}
$$

- The leading principle minors are

$$
det(a) = a; \quad det \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a^2; \quad det \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} = a(a^2 - 5)
$$

$\therefore V(x)$ is p.s.d. if $a \geq \sqrt{5}$, $\quad V(x)$ is p.d. if $a > \sqrt{5}$

- For n.d. the leading principle minors of
  - $-P$ should be positive. **OR**
  - P should alternate in sign with the first one neg. (odds: neg., even: pos.)

$\therefore V(x)$ is n.s.d. if $a \leq -\sqrt{5}$, $V(x)$ is n.d. if $a < -\sqrt{5}$,
$V(x)$ is sign indefinite for $-\sqrt{5} < a < \sqrt{5}$

Example 2

- Consider $\dot{x} = -g(x)$ where g(x) is locally Lip. on $(-a, a)$ & $g(0) = 0, \quad xg(x) > 0, \quad \forall x \neq 0, \quad x \in (-a, a)$. stability?
- origin is Equ. pt.
- Solution 1:
  - starting on either side of the origin will have to move toward the origin due to the sign of $\dot{x}$
  - $\therefore$ Origin is an isolated Equ. pt. and is asymptotically stable.
- Solution 2: using Lypunov theorem:
  - Consider the function $V(x) = \int_0^x g(y)dy$ over $D = (-a, a)$.
  - $V(x)$ is continuously differentiable, $V(0) = 0$ and $V(x) > 0, \ \forall x \neq 0 \rightsquigarrow$. V is a valid **Lyapunov candidate**
  - To see if it is really a Lyap. fcn, we have to take its derivative along system trajectory: $\dot{V}(x) = \frac{\partial V}{\partial x}(-g(x)) = -g^2(x) < 0, \quad \forall x \in D - \{0\}$
  - $\therefore$ $V(x)$ is a valid Lyap. fcn $\rightsquigarrow$ the origin is asymptotically stable.

Example 3: Frictionless Pendulum

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-g}{l} sinx_1 \end{aligned}$$

- Study stability of the Equ. pt. at the origin.
- A natural Lyap. fcn is the energy fcn:

$$V(x) = \frac{g}{l}(1 - cosx_1) + \frac{1}{2}x_2^2$$

- $V(0) = 0$ and $V(x)$ is p.d. over the domain $-2\pi \leq x_1 \leq 2\pi$.
- $\therefore \dot{V} = \frac{g}{l}\dot{x}_1 sinx_1 + x_2\dot{x}_2 = \frac{g}{l}x_2 sinx_1 - \frac{g}{l}x_2 sinx_1 = 0$
- $V(x)$ satisfies the condition of the Lyap. Theorem $\rightsquigarrow$ origin is **stable**

Example 4: Pendulum with Friction

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-g}{l} sinx_1 - \frac{k}{m} x_2 \end{aligned}$$

- Take the energy fcn as a Lyap. fcn candidate
$$V(x) = \frac{g}{l}(1 - cosx_1) + \frac{1}{2}x_2^2$$
- $\dot{V} = -\frac{k}{m}x_2^2$
- $\dot{V}(x)$ is n.s.d. It is not since n.d. since $\dot{V} = 0$ for $x_2 = 0$ and all $x_1 \neq 0$. the origin is only **stable**.
- But, phase portrait showed asymptotic stability!!
- Toward this end, let's choose:
$$V(x) = \frac{1}{2}x^T Px + \frac{g}{l}(1 - cosx_1)$$

- where $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ is p.d. $(P_{11} > 0,\ P_{22} > 0,\ P_{11}P_{22} - P_{12}^2 > 0)$

$$\begin{aligned} \dot{V} &= \frac{1}{2}(\dot{x}^T P x + x^T P \dot{x}) + \frac{g}{l}\dot{x}_1 sinx_1 = \frac{g}{l}(1 - P_{22})x_2 sinx_1 \\ &- \frac{g}{l}P_{12}x_1 sinx_1 + (P_{11} - P_{12}\frac{k}{m})x_1x_2 + (P_{12} - P_{22}\frac{k}{m})x_2^2 \end{aligned}$$

- Select P s.t. $\dot{V}$ is n.d. (cancel sign indefinite factors: $x_2 sinx_1$ and x_1x_2)
- $P_{22} = 1,\quad P_{11} = \frac{k}{m}P_{12},\quad 0 < P_{12} < \frac{k}{m}$ (for $V(x)$ to be p.d., take $P_{12} = \frac{1}{2}\frac{k}{m}$)

$$\therefore \dot{V} = -\frac{1}{2}\frac{g}{l}\frac{k}{m}x_1 sinx_1 - \frac{1}{2}\frac{k}{m}x_2^2$$

- $x_1 sinx_1 > 0 \ \ \forall\ 0 < |x_1| < \pi$, defining a domain D by $D = \{x \in R^2| \quad |x_1| < \pi\}$
- $\therefore$ $V(x)$ is p.d. and $\dot{V}$ is n.d. over D. Thus, origin is asymptotically stable (a.s.) by the theorem.

How Search for A Lyapunov Function?

- Lyapunov theorem is only **sufficient**.
- Failure of a Lyap. fcn candidate to satisfy the theorem **does not** mean the Equ. pt. is unstable.
- **Variable Gradient Method**
  - Idea is working backward:
    - Investigated an expression for $\dot{V}(x)$ and go back to choose the parameters of $V(x)$ so as to make $\dot{V}(x)$ n.d. (n.s.d)
  - Let $V = V(x)$ and $g(x) = \nabla_x V = \left(\frac{\partial V}{\partial x}\right)^T$
  - Then $\dot{V} = \frac{\partial V}{\partial x} f = g^T f$
  - Choose $g(x)$ s.t. it would be the gradient of a p.d. fcn V and make $\dot{V}$ n.d. (n.s.d)
  - $g(x)$ is the gradient of a scalar fcn iff the Jacobian matrix $\frac{\partial g}{\partial x}$ is symmetric:

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall i, j = 1, ..., n$$

Variable Gradient Method

- Select $g(x)$ s.t. $g^T(x)f(x)$ is n.d.
- Then, $V(x)$ is computed from the integral:

$$V(x) = \int_0^x g(y)dy = \int_0^x \sum_{i=1}^n g_i(y)dy_i$$

- The integration is taken over any path joining the origin to x. This can be done along the axes:

$$\begin{aligned} V(x) &= \int_0^{x_1} g_1(y_1, 0, ..., 0)dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, ..., 0)dy_2 \\ &+ ... + \int_0^{x_n} g_n(x_1, x_2, ..., y_n)dy_n \end{aligned}$$

- By leaving some parameters of g undetermined, one would try to choose them so that V is p.d.

Example 5:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - ax_2 \end{aligned}$$

where $a > 0$, $h(.)$ is locally Lip.,
$h(0) = 0,\ yh(y) > 0,\ \forall\ y \neq 0,\ y \in (-b, c),\ b, c > 0.$

- The pendulum is a special case of this system.
- Find proper Lypunov function?
- Applying variable gradient method, we must find $g(x)$ s.t. $\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$
- $\dot{V}(x) = g_1(x)x_2 - g_2(x)(h(x_1) + ax_2) < 0,\ \forall x \neq 0$ and
$$V(x) = \int_0^x g^T(y)dy > 0 \text{ for } x \neq 0$$
- Choose $g(x) = \begin{bmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2 \end{bmatrix}$ where $\alpha,\ \beta,\ \gamma,\ \delta$ to be determined

- To satisfy the symmetry req., we need

$$\beta(x) + \frac{\partial \alpha}{\partial x_2} x_1 + \frac{\partial \beta}{\partial x_2} x_2 = \gamma(x) + \frac{\partial \gamma}{\partial x_1} x_1 + \frac{\partial \delta}{\partial x_1} x_2$$

- $\dot{V}(x) =$
  $\alpha(x)x_1x_2 + \beta(x)x_2^2 - a\gamma(x)x_1x_2 - a\delta(x)x_2^2 - \delta(x)x_2h(x_1) - \gamma(x)x_1h(x_1)$
- To cancel the cross terms, let
  $\alpha(x)x_1 - a\gamma(x)x_1 - \delta(x)h(x_1) = 0$
- $\therefore \dot{V}(x) = -(a\delta(x) - \beta(x))x_2^2 - \gamma(x)x_1h(x_1)$
- For simplification, let $\delta(x) = \delta = cte,\ \gamma(x) = \gamma = cte,\ \beta(x) = \beta = cte$
- $\therefore \alpha(x)$ only depends on x_1
- symmetry is satisfied if $\beta = \gamma$.

$$g(x) = \begin{bmatrix} a\gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

- By integration, we get

$$\begin{aligned} V(x) &= \int_0^{x_1} (a\gamma y_1 + \delta h(y_1))dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2)dy_2 \\ &= \frac{1}{2}a\gamma x_1^2 + \delta \int_0^{x_1} h(y)dy + \gamma x_1 x_2 + \frac{1}{2}\delta x_2^2 \\ &= \frac{1}{2}x^T P x + \delta \int_0^{x_1} h(y)dy \end{aligned}$$

- where $P = \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}$.
- Choosing $\delta > 0,\ 0 < \gamma < a\delta \Longrightarrow V$ is p.d. & $\dot{V}$ is n.d.
- e.g., taking $\gamma = ak\delta, \quad 0 < k < 1$ yields

  $V(x) = \frac{\delta}{2}x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)dy$
- over $D = \{x \in R^n | -b < x_1 < c\}$ conditions of the theorem are satisfied

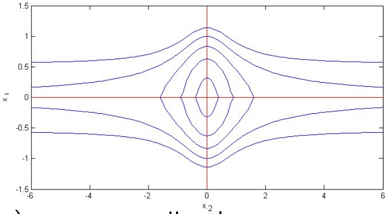
  $\rightsquigarrow$ $x = 0$ is asymptotically stable.

Region of Attraction

- For asymptotically stable Equ. pt.:
  How far from the origin can the trajectory be and still converges to the origin as $t \longrightarrow \infty$?
- Let $\phi(t, x)$ be the sol. of $\dot{x} = f(x)$ starting at x_0.
  Then, the Region of Attraction (RoA) is defined as
  the set of all pts. x s.t. $\lim_{t \longrightarrow \infty} \phi(t, x) = 0$
- Lyap. fcn can be used to estimate the RoA:
  - If there is a Lyap. fcn. satisfying asymptotic stability over domain D,
  - and set $\Omega_c = \{x \in R^n | V(x) \leq c\}$ is bounded and contained in D
  - $\therefore$ all trajectories starting in Ω_c remains there and converges to 0 at $t \to \infty$.
- Under what condition the RoA be R^n(i.e., the Equ. pt. is globally asymptotically stable (g.a.s))?
  - the conditions of stability theory must hold globally, i.e. $D = R^n$.
  - This not enough!
  - for large c, the set Ω_c should be kept bounded.
    i.e., reduction of $V(x)$ should also result in reduction of $\|x\|$.

Region of Attraction

- **Example:** $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$



  - It's clear that $V(x)$ can get smaller, but x grows unboundedly

- **Babashin-Krasovskii Theorem:** *Let $x = 0$ be an Equ. pt. of $\dot{x} = f(x)$. Let $V : R^n \longrightarrow R$ be a continuously differentiable fcn. s.t.:*
  - $V(0) = 0$
  - $V(x) > 0, \ \forall x \neq 0$
  - $\|x\| \longrightarrow \infty \implies V(x) \longrightarrow \infty$ (i.e. it is **radially unbounded**)
  - $\dot{V} < 0, \quad \forall x \neq 0$

  then $x = 0$ is globally asymptotically stable

Example 6 : Globally Asymptotically Stable

- Reconsider Example 5 ($h(0) = 0, \quad xh(x) > 0, \quad \forall x \neq 0, \quad x \in (-a, a)$)
- but assume that $xh(x) > 0$ hold for **all** $x \neq 0$.
  - The Lyap. fcn:

$$V(x) = \frac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)dy$$

    is p.d. $\forall x \in R^2$
  - $V(x)$ is radially unbounded.
  - $\dot{V} = -a\delta(1-k)x_2^2 - ak\delta x_1 h(x_1) < 0, \quad \forall x \in R^2$
  - $\therefore$ origin is **g.a.s.**
- **Important:** Since the origin is **g.a.s.**, then it must be the unique Equ. pt. of the system
- **g.a.s.** is not satisfied for multiple Equ. pt. problem such as pendulum.

Instability Theorem

- **Chetaev's Theorem** *Let $x = 0$ be an Equ. pt. of $\dot{x} = f(x)$. Let $V : D \longrightarrow R$ be a continuously differentiable fcn such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrary small $\|x_0\|$. Define a set $\nu = \{x \in B_r | V(x) > 0\}$ where $B_r = \{x \in R^n | \|x\| < r\}$ and suppose that $\dot{V}(x)$ is p.d. in ν. Then, $x = 0$ is unstable.*
- **Example:**

$$\begin{aligned} \dot{x}_1 &= x_1 + g_1(x) \\ \dot{x}_2 &= -x_2 + g_2(x) \end{aligned}$$

  where $|g_i(x)| \leq k\|x\|_2^2$ in a neighborhood D of origin
- The inequality implies, $g_i(0) = 0 \Longrightarrow$ origin is an Equ. pt.
- Consider: $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$
  - On the line $x_2 = 0$, $V(x) > 0$.
  - $\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$
  - Since $|x_1 g_1(x) - x_2 g_2(x)| \leq \sum_{i=1}^{2} |x_i||g_i(x)| \leq 2k\|x\|_2^3$
  - $\therefore \dot{V}(x) \geq \|x\|_2^2 - 2k\|x\|_2^3 = \|x\|_2^2(1 - 2k\|x\|_2)$
  - Choosing r s.t. $B_r \subset D$ and $r < \frac{1}{2}k \Longrightarrow$ origin is **unstable**.

Invariance Principle

- Recall the pendulum example:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{-g}{l} sinx_1 - \frac{k}{m} x_2$$

  $\dot{V}(x) = \frac{-k}{m} x_2^2$ which is n.s.d.
- $\therefore$ Lyap. theorem shows **only stability**. However,
  - $\dot{V}$ is negative everywhere except at $x_2 = 0$ where $\dot{V} = 0$.
  - To get $\dot{V} = 0$, the trajectory must be confined to $x_2 = 0$
  - Now, from the model $x_2(t) \equiv 0 \implies \dot{x}_1 \equiv 0 \implies x_1(t) = cte$ and $x_2(t) \equiv 0 \implies \dot{x}_2 \equiv 0 \implies sinx_1 \equiv 0$
  - Hence, on the segment $-\pi < x_1 < \pi$ of $x_2 = 0$ line, the system can maintain $\dot{V} = 0$ only at $x = 0$.
- Therefore, $V(x)$ decrease to zero and $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$.
- The idea follows from **LaSalle's Invariance Principle**

Invariance Principle

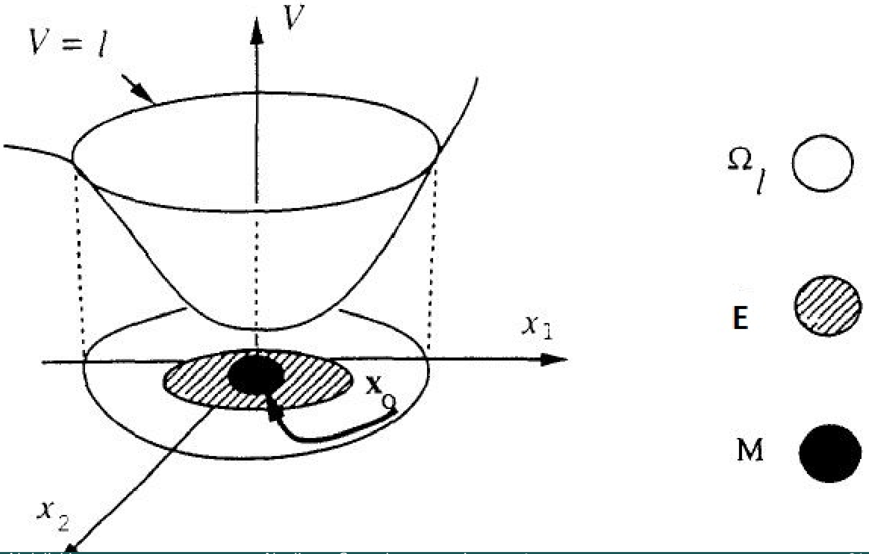
- A set M is said to be a positively invariant set with respect to $\dot{x} = f(x)$, if $x(0) \in M \Longrightarrow x(t) \in M, \ \forall t \geq 0$
  - If a solution belongs to M at some time instant, then it belongs to M for all future time.
- $\therefore$ Equ. points and limit cycle are invariant sets
- Also the set $\Omega = \{x \in R^n | V(x) \leq c\}$ with $\dot{V} \leq 0 \ \ \forall x \in \Omega$ is a positively invariant set.

Lasalle's Theorem:

- *Let Ω be a compact set with property that every solution of $\dot{x} = f(x)$ starting in Ω remains in Ω for all future time.*
  - *Let $V : \Omega \longrightarrow R$ be a continuously differentiable fcn s.t. $\dot{V}(x) \leq 0$ in Ω.*
  - *Let E be the set of all pts in Ω where $\dot{V}(x) = 0$*
  - *Let M be the largest invariant set in E.*

  Then, every sol. starting in Ω approaches M as $t \longrightarrow \infty$
- Unlike Lyap. theorem, Lasalle's theorem **does not** require $V(x)$ to be **p.d**
- Only Ω should be bounded
  - If V is p.d. $\Rightarrow$ $\Omega = \{x \in R^n | V(x) \leq c\}$ is bounded for suff. small c
  - If V is radially unbounded $\Rightarrow$ Ω is bounded for all c no matter V is p.d. or not
- To show **a.s.** of the origin $\longrightarrow$ show largest invariant set in E is the origin.
- $\therefore$ Show that no solution can stay forever in E other than $x = 0$.

Lasalle's Theorem:



Barbashin and Krasovskii Corollaries

- **Corollary 1:** *Let $x = 0$ be an Equ. pt of $\dot{x} = f(x)$. Let $V : D \to R$ be a continuously differentiable p.d. fcn on a domain D containing the origin $x = 0$, s.t. $\dot{V}(x) \leq 0$ in D. Let $S = \{x \in D | \dot{V} = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) = 0$. Then, the origin is* **a.s.**
- **Corollary 2:** *Let $x = 0$ be an Equ. pt. of $\dot{x} = f(x)$. Let $V : R^n \longrightarrow R$ be a continuously differentiable, radially unbounded, p.d. fcn s.t. $\dot{V}(x) \leq 0 \ \ \forall x \ \in \ R^n$. Let $S = \{x \in R^n | \dot{V} = 0\}$ and suppose that no solution can stay in S forever except $x = 0$. Then, the origin is* **g.a.s.**

Example 6:

- Consider
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -g(x_1) - h(x_2)\end{aligned}$$

where $g(.)$ & $h(.)$ are locally Lip. and satisfy
$$\begin{aligned}g(0) &= 0, \quad yg(y) > 0 \quad \forall y \neq 0, \ y \in (-a, a)\\ h(0) &= 0, \quad yh(y) > 0 \quad \forall y \neq 0, \ y \in (-a, a)\end{aligned}$$

- The system has an isolated Equ. pt. at origin. Let
$$V(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} g(y)dy$$

  - $D = \{x \in R^2 | -a < x_i < a\} \implies V(x) > 0$ in D
  - $\dot{V} = g(x_1)x_2 + x_2(-g(x_1) - h(x_2)) = -x_2 h(x_2) \leq 0$
  - Thus, $\dot{V}$ is n.s.d. and the origin is stable by Lyap. theorem

Example 6:

- Using Lasalle's theorem, define $S = \{x \in D | \dot{V} = 0\}$
  - $\dot{V} = 0 \implies x_2 h(x_2) = 0 \implies x_2 = 0$, since $-a < x_2 < a$
  - Hence $S = \{x \in D | x_2 = 0\}$. Suppose $x(t)$ is a traj. $\in S \; \forall t$
  - $\therefore x_2(t) \equiv 0 \implies \dot{x}_1 \equiv 0 \implies x_1(t) = c$, where $c \in (-a, a)$. Also $x_2(t) \equiv 0 \implies \dot{x}_2 \equiv 0 \implies g(c) = 0 \implies c = 0$
- $\therefore$ Only solution that can stay in $S \; \forall \; t \geq 0$ is the origin $\implies x = 0$ is **a.s.**
- Now, Let $a = \infty$ and assume g satisfy:
  $$\int_0^y g(z)dz \longrightarrow \infty \text{ as } |y| \longrightarrow \infty.$$
- The Lyap. fcn $V(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} g(y)dy$ is **radially unbounded**.
- $\dot{V} \leq 0$ in R^2 and note that $S = \{x \in R^2 | \dot{V} = 0\} = \{x \in R^2 | x_2 = 0\}$ contains no solution other than origin $\implies x = 0$ is **g.a.s.**

Summary

- Lyapunov Direct Method: The origin of an autonomous system $\dot{x} = f(x)$ is stable if there is a continuously differentiable, p.d. function $V(x)$ s.t. $\dot{V}(x)$ is n.s.d., and] is asymptotically if $\dot{V}(x)$ is n.d.
  - $V(x)$ is p.d./p.s.d. iff $\lambda_i\{P\} > 0/\lambda_i\{P\} \geq 0$ iff all leading principle minors of P are positive / non-negative.
  - Variable Gradient Method: To find a Lyap fcn: Choose $g(x)$ s.t.
    1. $\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}$
    2. $g^T(x)f(x)$ is n.d. (n.s.d)
    3. $V(x) = \int_0^{x_1} g_1(y_1, 0, ..., 0)dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, ..., 0)dy_2 + ... + \int_0^{x_n} g_n(x_1, x_2, ..., y_n)dy_n$ is P.d.
  - **Babashin-Krasovskii Theorem:** *Let $x = 0$ be an Equ. pt. of $\dot{x} = f(x)$. Let $V : R^n \longrightarrow R$ be a continuously differentiable fcn. s.t.: $V(0) = 0$, $V(x) > 0$, $\forall x \neq 0$,* $\|x\| \longrightarrow \infty \implies V(x) \longrightarrow \infty$ (i.e. it is **radially unbounded**), $\dot{V} < 0$, $\forall x \neq 0$ *then $x = 0$ is globally asymptotically stable*

Summary

- Another method for study a.s is defined based on Lasalle Theorem:
- **Corollary 1:** *Let $x = 0$ be an Equ. pt of $\dot{x} = f(x)$. Let $V : D \to R$ be a continuously differentiable p.d. fcn on a domain D containing the origin $x = 0$, s.t. $\dot{V}(x) \leq 0$ in D. Let $S = \{x \in D | \dot{V} = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) = 0$. Then, the origin is* **a.s.**
  - The origin is **g.a.s.** if $D = R^n$, and $V(x)$ is radially unbounded.

Invariance Principle

- Lasalle's theorem can also extend the Lyap. theorem in three different directions
  1. It gives an estimate of the **RoA** not necessarily in the form of $\Omega_c = \{x \in R^n | \ V(x) \leq c\}$. The set can be any **positively invariant set** which leads to less conservative estimate.
  2. Can determine stability of **Equ. set**, rather than **isolated Equ. pts.**
  3. The function $V(x)$ **does not** have to be **positive definite**.
- **Example 7:** shows how to use Lasalle's theorem for system with Equ. sets rather than isolated Equ. pts
  - A simple adaptive control problem:

$$\dot{x} = ax + u \qquad a \text{ unknown}$$

with the adaptive control law

$$u = -kx; \quad \dot{k} = \gamma x^2, \quad \gamma > 0$$

Example 7:

- Let $x_1 = x$, $x_2 = k$, we get:

$$\begin{aligned}\dot{x}_1 &= -(x_2 - a)x_1\\ \dot{x}_2 &= \gamma x_1^2\end{aligned}$$

- The line $x_1 = 0$ is an Equ. set
- Show that the traj. of closed-loop system approaches this set as $t \longrightarrow \infty$
  - i.e. the adaptive system regulates y to zero ($x_1 \longrightarrow 0$ as $t \longrightarrow \infty$).
- Consider the Lyap. fcn candidate:

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2$$

where $b > a$.

- $\dot{V} = -x_1^2(b - a) \leq 0$
- The set $\Omega = \{x \in R^n | V(x) \leq c\}$ is a compact **positively invariant set**

Example 7:

- $V(x)$ is radially unbounded $\Longrightarrow$ Lasalle's theorem conditions are satisfied with the set E as $$E = \{x \in \Omega | x_1 = 0\}$$
- Since any pt on $x_1 = 0$ line is an Equ. pt, E is an invariant set: $M = E$.
- Hence, every trajectory starting in Ω approaches E as $t \longrightarrow \infty$, i.e. $x_1(t) \longrightarrow 0$ as $t \longrightarrow \infty$.
- V is radially unbounded $\Longrightarrow$ the result is global
- Note that in the above example the Lyapunov function depends on a constant b which is required to satisfy $b > a$
- But it is not known $\rightsquigarrow$ we may not know the constant b explicitly, but we know that it always exists.
- This highlights another feature of Lyapunov's method:
  - In some situations, we may be able to assert the existence of a Lyapunov function that satisfies the conditions, even though we may not explicitly know that function.

Linear System and Linearization

- Given $\dot{x} = Ax$, the Equ. pt. is at origin
- It is isolated **iff** det $A \neq 0$,
- System has an Equ. subspace if det $A = 0$, the subspace is the null space of A.
- The linear system cannot have multiple isolated Equ. pt. since
  - Linearity requires that if x_1 and x_2 are Equ. pts., then all pts. on the line connecting them should also be Equ. pts.
- **Theorem:** *The Equ. pt. $x = 0$ of $\dot{x} = Ax$ is stable* **iff** *all eigenvalues of A satisfy $Re\{\lambda_i\} \leq 0$ and every eigenvalue with $Re\{\lambda_i\} = 0$ and algebric multiplicity $q_i \geq 2$, $rank(A - \lambda_i I) = n - q_i$, where n is dimension of x. The Equ. pt. $x = 0$ is globally asymptotically stable* **iff** $Re\lambda_i < 0$.

Linear Systems and Linearization

- When all eigenvalues of A satisfy $Re\lambda_i < 0$, A is called a Hurwitz matrix.
- Asymptotic stability can be verified by using Lyapunov's method :
  - Consider a quadratic Lyap. fcn candidate:

$$\begin{aligned} V(x) &= x^T \ P \ x, \qquad P = P^T > 0 \\ \dot{V} &= \dot{x}^T P x + x^T P \dot{x}, \ \ x^T (A^T P + PA) x \triangleq -x^T \ Q \ x \end{aligned}$$

where

$$A^T \ P + P \ A = -Q, \quad Q = Q^T \quad \textbf{Lyapunov Equation}$$

  - If Q is p.d., then we conclude that $x = 0$ is **g.a.s.**
  - We can proceed alternatively as follows:
  - Start by choosing $Q = Q^T$, $Q > 0$, then solve the Lyap. eqn. for P.
  - If $P > 0$, then $x = 0$ is **g.a.s.**

Linear Systems and Linearization

- **Theorem:** *A matrix A is a stable matrix, i.e. Re $\lambda_i < 0$ **iff** for every given $Q = Q^T > 0$, $\exists\ P = P^T > 0$ that satisfies the Lyap. eq. Moreover, if A is a stable matrix, then P is unique.*
- **Example 8:**

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

  - Let $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Q^T > 0$
  - denote $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = P^T > 0$
  - The Lyap. eq. $A^T\ P + P\ A = -Q$ becomes

$$\begin{aligned} 2\ P_{12} &= -1 \\ -P_{11} - P_{12} + P_{22} &= 0 \\ -2\ P_{12} - 2P_{22} &= -1 \end{aligned}$$

Linear Systems and Linearization

$$\begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{12} \\ P_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \implies \begin{bmatrix} P_{11} \\ P_{12} \\ P_{22} \end{bmatrix} = \begin{bmatrix} 1.5 \\ -.5 \\ 1 \end{bmatrix} \tag{1}$$

- Let $P = P^T = \begin{bmatrix} 1.5 & -.5 \\ -.5 & 1 \end{bmatrix} > 0 \implies x = 0$ is **g.a.s**
- **Remark:** Computationally, there is no advantages in computing the eigenvalues of A over solving Lyap. eqn.

Linear Systems and Linearization

- Consider $\dot{x} = f(x)$ where $f : D \longrightarrow R^n$, $D \subset R^n$, is continuously diff. Let $x = 0$ is in the interior of D and $f(0) = 0$.
- In a small neighborhood of $x = 0$, the nonlinear system $\dot{x} = f(x)$ can be linearized by $\dot{x} = Ax$.
  - Proof: Ref. Khalil's book
- **Theorem (Lyapunov's First Method):**
- *Let $x = 0$ be an Equ. pt. for $\dot{x} = f(x)$ where $f : D \longrightarrow R^n$ is continuously differentiable and D is a nghd of origin. Let $A = \frac{\partial f}{\partial x}\Big|_{x=0}$, then*
  1. *$x = 0$ is* **a.s.** *if $Re\lambda_i < 0, \quad i = 1, ..., n$*
  2. *$x = 0$ is* **unstable** *if $Re\lambda_i > 0$, for one or more eigenvalues*

Linear Systems and Linearization

- **Example 9:** $\dot{x} = ax^3$
  - Linearization about $x = 0$ yields:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.3ax^2\right|_{x=0} = 0$$

  - Linearization fails to determine stability
  - If $a < 0$, $x = 0$ is **a.s.**
  - To see this, let $V(x) = x^4 \Longrightarrow \dot{V} = 4x^3\dot{x} = 4ax^6$
  - If $a > 0$, $x = 0$ is **unstable**
  - If $a \leq 0$, $x = 0$ is **stable**, starting at any x, remains in x
- **Example 10:**

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\left(\frac{g}{l}\right) sin\ x_1 - \left(\frac{k}{m} x_2\right)
\end{aligned}$$

Linear Systems and Linearization

- Linearization about 2 Equ. pts. $(0,0)$ & $(\pi,0)$:

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} cos\ x_1 & -\frac{k}{m} \end{bmatrix}$$

- At $(0,0)$
  - $A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$, $\rightsquigarrow \lambda_{1,2} = -\frac{1}{2}\frac{k}{m} \pm \frac{1}{2}\sqrt{(\frac{k}{m})^2 - 4\frac{g}{l}}$
  - $\therefore\ \forall g,k,l,m\ > 0 \Longrightarrow\ Re(\lambda_1,\lambda_2) < 0 \Longrightarrow\ x = 0$ is **a.s.**
  - If $k = 0\ \Longrightarrow\ Re(\lambda_1,\lambda_2) = 0 \Longrightarrow$ eigenvalues on $j\omega$ axis, $rank(A - \lambda_i I) \neq 0$ $\therefore$ stability cannot be determined.
- At $(\pi,0)\rightsquigarrow$, change the variable to $z_1 = x_1 - \pi$, $z_2 = x_2$
  - $A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}$, $\rightsquigarrow \lambda_{1,2} = -\frac{1}{2}\frac{k}{m} \pm \frac{1}{2}\sqrt{(\frac{k}{m})^2 + 4\frac{g}{l}}$
  - $\therefore\ \forall g,k,l,m\ > 0 \Longrightarrow$ there is one eigenvalue in the open right-half plane $\Longrightarrow\ x = 0$ is **unstable**.

Example: Robot Manipulator

- Dynamics:
$$M(q)\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + B\dot{\mathbf{q}} + g(\mathbf{q}) = u \tag{2}$$
where $M(q)$ is the $n \times n$ inertia matrix of the manipulator
- $C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ is the vector of Coriolis and centrifugal forces
- $g(\mathbf{q})$ is the term due to the Gravity
- $B\dot{\mathbf{q}}$ is the viscous damping term
- u is the input torque, usually provided by a DC motor
- **Objective**: To regulate the joint position q around desired position q_d.
- A common control strategy PD+Gravity:
$$u = K_P\tilde{\mathbf{q}} - K_D\dot{\mathbf{q}} + g(\mathbf{q})$$
where $\tilde{\mathbf{q}} = \mathbf{q}_d - \mathbf{q}$ is the error between the desired and actual position
- K_P and K_D are diagonal positive proportional and derivative gains.

Example: Robot Manipulator

- Consider the following Lyap. fcn candidate:

$$V = \frac{1}{2}\dot{\mathbf{q}}^T M(\mathbf{q})\dot{\mathbf{q}} + \frac{1}{2}\tilde{\mathbf{q}}^T K_P \tilde{\mathbf{q}}$$

  - The first term is the kinetic energy of the robot and the second term accounts for “artificial potential energy” associated with virtual spring in PD control law (proportional feedback $K_p\tilde{\mathbf{q}}$)
  - Physical properties of a robot manipulator:
    1. The inertia matrix $M(q)$ is positive definite
    2. The matrix $\dot{M}(q) - 2C(\mathbf{q}, \dot{\mathbf{q}})$ is skew symmetric
  - V is positive in R^n except at the goal position $\mathbf{q} = \mathbf{q}^d, \quad \dot{\mathbf{q}} = 0$

$$\dot{V} = \dot{\mathbf{q}}^T M(q)\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^T \dot{M}(q)\dot{\mathbf{q}} - \dot{\mathbf{q}}^T K_P \tilde{\mathbf{q}}$$

  - Substituting $M(q)\ddot{\mathbf{q}}$ from (2) into the above equation yields

$$\begin{aligned}\dot{V} &= \dot{\mathbf{q}}^T(u - C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} - B\dot{\mathbf{q}} - g(\mathbf{q})) + \frac{1}{2}\dot{\mathbf{q}}^T \dot{M}(q)\dot{\mathbf{q}} - \dot{\mathbf{q}}^T K_P \tilde{\mathbf{q}} \\ &= \dot{\mathbf{q}}^T(u - B\dot{\mathbf{q}} - K_P\tilde{\mathbf{q}} - g(\mathbf{q})) + \frac{1}{2}\dot{\mathbf{q}}^T(\dot{M}(q) - 2C(\mathbf{q}, \dot{\mathbf{q}}))\dot{\mathbf{q}}\end{aligned}$$

Example: Robot Manipulator

$$\dot{V} = \dot{\mathbf{q}}^T(u - B\dot{\mathbf{q}} - K_P\tilde{\mathbf{q}} - g(\mathbf{q}))$$

- where $\dot{M} - 2C$ is skew symmetric $\rightsquigarrow$ $\dot{\mathbf{q}}^T(\dot{M}(q)\dot{\mathbf{q}} - C(\mathbf{q}, \dot{\mathbf{q}}))\dot{\mathbf{q}} = 0$
- Substitute PD control law for u, we get:

$$\dot{V} = -\dot{\mathbf{q}}^T(K_D + B)\dot{\mathbf{q}} \leq 0 \tag{3}$$

- The goal position is stable since V is non-increasing
- Use the invariant set theorem:
  - Suppose $\dot{V} \equiv 0$, then (3) implies that $\dot{\mathbf{q}} \equiv 0$ and hence $\ddot{\mathbf{q}} \equiv 0$
  - From Equ. of motion (2) with PD control, we have

$$M(q)\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + B\dot{\mathbf{q}} = K_P\tilde{\mathbf{q}} - K_D\dot{\mathbf{q}}$$

    we must then have $0 = K_P\tilde{\mathbf{q}}$ which implies that $\tilde{\mathbf{q}} = 0$
  - V is radially unbounded.
  - $\therefore$ Global asymptotic stability is ensured.

Example: Robot Manipulator

- In case, the gravitational terms is not canceled, $\dot{V}$ is modified to:

$$\dot{V} = -\dot{\mathbf{q}}^T((K_D + B)\dot{\mathbf{q}} + g(q))$$

- The presence of gravitational term means PD control alone cannot guarantee asymptotic tracking.
- Assuming that the closed loop system is stable, the robot configuration $\mathbf{q}$ will satisfy

$$K_P(\mathbf{q}_d - \mathbf{q}) = g(\mathbf{q})$$

- The physical interpretation of the above equation is that:
  - The configuration q must be such that the motor generates a steady state "holding torque" $K_P(\mathbf{q}_d - \mathbf{q})$ sufficient to balance the gravitational torque $g(\mathbf{q})$.
- $\therefore$ the steady state error can be reduced by increasing K_P.

Control Design Based on Lyapunov's Direct Method

- Basically there are two approaches to design control using Lyapunov's direct method
  - Choose a control law, then find a Lyap. fcn to justify the choice
  - Candidate a Lyap. fcn, then find a control law to satisfy the Lyap. stability conditions.
- Both methods have a trial and error flavor
- In robot manipulator example the first approach was applied:
  - First a PD controller was chosen based on physical intuition
  - Then a Lyap. fcn. is found to show g.a.s.

Control Design Based on Lyapunov's Direct Method

- Example: Regulator Design
- Consider the problem of stabilizing the system:
$$\ddot{x} - \dot{x}^3 + x^2 = u$$
- In other word, make the origin an asymptotically stable Equ. pt.
- Recall the example:
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -g(x_1) - h(x_2)\end{aligned}$$

where $g(.)$ & $h(.)$ are locally Lip. and satisfy
$$\begin{aligned}g(0) &= 0, \quad yg(y) > 0 \;\; \forall y \neq 0, \;\; y \in (-a, a)\\ h(0) &= 0, \quad yh(y) > 0 \;\; \forall y \neq 0, \;\; y \in (-a, a)\end{aligned}$$

- Asymptotic stability of such system could be shown by selecting the following Lyap. fcn:
$$V(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} g(y)dy$$

Example: Regulator Design

- Let $x_1 = x,\ x_2 = \dot{x}$. The above example motivates us to select the control law u as

$$u = u_1(\dot{x}) + u_2(x)$$

where

$$\dot{x}(\dot{x}^3 + u_1(\dot{x})) < 0 \quad \text{for} \quad \dot{x} \neq 0$$
$$x(u_2(x) - x^2) < 0 \quad \text{for} \quad x \neq 0$$

- The globally stabilizing controller can be designed even in the presence of some uncertainties on the dynamics:

$$\ddot{x} + \alpha_1 \dot{x}^3 + \alpha_2 x^2 = u$$

where α_1 and α_2 are unknown, but s.t. $\alpha_1 > -2$ and $|\alpha_2| < 5$

- This system can be globally stabilized using the control law:

$$u = -2\dot{x}^3 - 5(x + x^3)$$

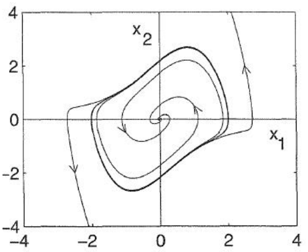
Estimating Region of Attraction

- Sometimes just knowing a system is a.s. is not enough. At least an estimation of RoA is required.
  - Example: Occurring fault and finding "critical clearance time"
- Let $x = 0$ be an Equ. pt. of $\dot{x} = f(x)$. Let $\phi(t, x)$ be the sol starting at x at time t=0. The Region Of Attraction (RoA) of the origin denoted by R_A is defined by:

$$R_A = \{x \in R^n | \phi(t, x) \longrightarrow 0 \text{ as } t \longrightarrow \infty\}$$

- **Lemma:** *If $x = 0$ is an a.s. Equ. pt. of $\dot{x} = f(x)$, then its RoA R_A is an open, connected, invariant set. Moreover, the boundary of RoA, ∂R_A, is formed by trajectories of $\dot{x} = f(x)$.*
- $\therefore$ one way to determine RoA is to characterize those trajectories that lie on ∂R_A.

Example: Van-der-Pol



- Dynamics of oscillator in reverse time

$$\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2\end{aligned}$$

- The system has an Equ. pt at origin and an unstable limit cycle.
- The origin is a stable focus $\longrightarrow$ it is **a.s.**

Example: Van-der-Pol

- Checking by linearizaton method

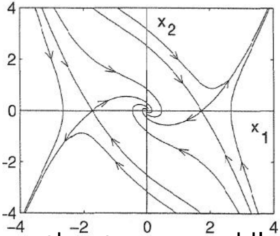
$$A = \frac{\partial f}{\partial x}\bigg|_{x=0} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

- $\lambda = -1/2 \pm j\sqrt{3}/2 \rightsquigarrow Re\ \lambda_i \ < 0$
- Clearly, RoA is bounded since trajectories outside the limit cycle drift away from it
- $\therefore\ \partial R_A$ is the limit cycle

Example 11:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - x_2$$

- There are 3 isolated Equ. pts. $(0,0)$, $(\sqrt{3},0)$, $(-\sqrt{3},0)$.



- $(0,0)$ is a stable focus, the other two are saddle pts.
- $\therefore$ Origin is **a.s.** and other two are unstable (follows from linearization).
- stable trajectories of the saddle points form two separatrices that are ∂R_A
- RoA is unbounded

Example 11:

- Recall the example:

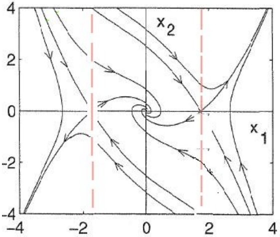
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -h(x_1) - ax_2\\ V &= \frac{\delta}{2}x^T\begin{bmatrix} ka^2 & ka\\ ka & 1\end{bmatrix}x + \delta\int_0^{x_1} hdy\end{aligned}$$

- Let

$$V = \tfrac{1}{2}x^T\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix}x + \int_0^{x_1}\left(y - \tfrac{1}{3}y^3\right)dy$$

$$= \tfrac{3}{4}x_1^2 - \tfrac{1}{12}x_1^4 + \tfrac{1}{2}x_1x_2 + \tfrac{1}{2}x_2^2$$

- We get: $\dot{V} = -\frac{1}{2}x_1^2\left(1 - \frac{1}{3}x_1^2\right) - \frac{1}{2}x_2^2$
- Define $D = \{x \in R^2 \mid -\sqrt{3} < x_1 < \sqrt{3}\}$
- $\therefore$ $V(x) > 0$ and $\dot{V}(x) < 0$ in $D - \{0\}$,
- From the phase portrait $\Longrightarrow$ D is not a subset of R_A. Tell me why?!!



Estimating RoA

- Traj starting in D move from one Lyap. surface to $V(x) = c_1$ to an inner surface $V(x) = c_2$ with $c_2 < c_1$.
- However, there is no guarantee that the traj. will remain in D forever.
- Once, the traj leaves D, no guarantee that $\dot{V}$ remains negative.
- This problem does not occur in R_A since R_A is an invariant set.
- The simplest estimate is given by the set

$$\Omega_c = \{x \in R^n | V(x) \leq c\}$$

  where Ω_c isbounded and connected and $\Omega_c \in D$
- Note that $\{V(x) \leq c\}$ may have more than one component, only the bounded component which belong to D is acceptable.
  - Example: If $V(x) = x^2/(1 + x^4)$.and $D = \{|x| < 1\}$
  - The set $\{V(x) \leq 1/4\}$ has two components $\{|x| \leq \sqrt{2 - \sqrt{3}}\}$ and $\{|x| \leq \sqrt{2 + \sqrt{3}}\} \rightsquigarrow$ only $\{|x| \leq \sqrt{2 - \sqrt{3}}\}$ is acceptable.

Estimating RoA

- To find RoA, first we need to find a domain D in which $\dot{V}$ is n.d.
- Then, a bounded set $\Omega_c \subset D$ shall be sought
- We are interested in largest set Ω_c, i.e. the largest value of c since Ω_c is an estimate of R_A.
- V is p.d. everywhere in R^2.
- If $V(x) = x^T P x$, let $D = \{x \in R^2 | \; \|x\| \leq r\}$. Once, D is obtained, then select $\Omega_c \subset D$ by $c < \min_{\|x\|=r} V(x)$
- In words, the smallest $V(x) = c$ which fits into D.
- Since
$$x^T P x \geq \lambda_{min}(P)\|x\|^2$$
- We can choose
$$c < \lambda_{min}(P) r^2$$
- To enlarge the estimate of $R_A \implies$ find largest ball on which $\dot{V}$ is n.d.

Example 12:

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2$$

- From the linearization $\frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ is stable
- Taking $Q = I$ and solve the Lyap. equation:
  $PA + A^T P = -I \implies P = \begin{bmatrix} 1.5 & -.5 \\ -.5 & 1 \end{bmatrix}$
- $\lambda_{min}(P) = 0.69$
- $\dot{V} = -(x_1^2 + x_2^2) - (x_1^3 x_2 - 2x_1^2 x_2^2) \leq -\|x\|_2^2 + |x_1||x_1 x_2||x_1 - 2x_2| \leq -\|x\|_2^2 + \frac{\sqrt{5}}{2}\|x\|_2^4$
  - where $|x_1| \leq \|x\|_2$, $|x_1 x_2| \leq \|x\|_2^2/2$, $|x1 - 2x_2| \leq \sqrt{5}\|x\|_2$
- $\dot{V}$ is n.d. on a ball D of radius
  $r^2 = 2/\sqrt{5} = 0.894 \rightsquigarrow c < 0.894 \times 0.69 = 0.617$

Example 12:

- To find less conservative estimate of Ω_c:
- Let $x_1 = \rho \; cos\theta, \quad x_2 = \rho \; sin\theta$

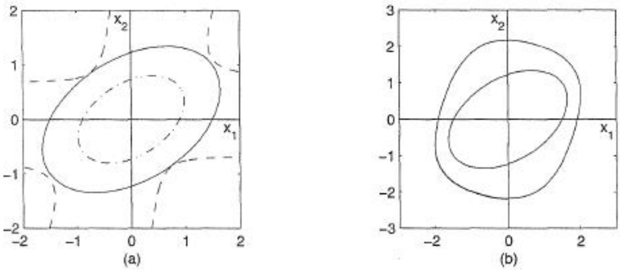
$$\begin{aligned} \dot{V} &= -\rho^2 + \rho^4 cos^2\theta sin\theta \left(2sin\theta - cos\theta\right) \\ &\le -\rho^2 + \rho^4 \left|cos^2\theta sin\theta\right| \left|2sin\theta - cos\theta\right| \\ &\le -\rho^2 + \rho^4 (.3849)(2.2361) \\ &\le -\rho^2 + .861\rho^4 < 0 \;\; \text{for } \rho^2 < \frac{1}{.861} \end{aligned}$$

- $c = .8 < \frac{.69}{.861} = .801$
- Thus the set:

$$\Omega_c = \{x \in R^2 |\; V(x) \;\le\; .8\} \text{ is an estimate of } R_A.$$

Example 12:

- A lesser conservative estimation of RoA:
  - plot the contour of $\dot{V} = 0$
  - plot $V(x) = c$ for increasing c to find largest c where $\dot{V} < 0$
- The c obtained by this method is $c = 2.25$.



(a) Contours of $\dot{V}(x) = 0$ (dashed), $V(x) = 0.8$ (dash-dot), and $V(x) = 2.25$ (solid) (b) comparison of the region of attraction with its estimate.

Example 13:

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_1x_2\\ \dot{x}_2 &= -x_2 + x_1x_2\end{aligned}$$

- There are two Equ. pts., $(0,0)$, $(1,2)$.
- At $(1,2) \rightsquigarrow A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \Longrightarrow$ unstable $\left(\lambda_{1,2} = \pm\sqrt{2}\right)$ (saddle pt.)
- At $(0,0) \rightsquigarrow A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \Longrightarrow$ **a.s.**
- Taking $Q = I$ and solving Lyap Eq. $A^TP + PA = -I \Rightarrow P = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$
- $\therefore$ The Lyap. fcn is $\quad V(x) = x^TPx$
- We have $\dot{V} = -(x_1^2 + x_2^2) + \frac{1}{2}(x_1^2x_2 + 2x_1x_2^2)$
- Find largest D s.t. $\dot{V}$ is n.d. in D.
- Let $x_1 = \rho\ cos\theta, \quad x_2 = \rho\ sin\theta$

Example 13:

$$\begin{aligned}\dot{V} &= -\rho^2 + \rho^3 cos\theta sin\theta \left(sin\theta + \frac{1}{2} cos\theta \right) \\ &\leq -\rho^2 + \frac{1}{2}\rho^3 \left| sin2\theta \right| \left| sin\theta + \frac{1}{2} cos\theta \right| \\ &\leq -\rho^2 + \frac{\sqrt{5}}{4}\rho^3 < 0 \ \text{ for } \rho < \frac{4}{\sqrt{5}}\end{aligned}$$

- Since $\lambda_{min}(P) = \frac{1}{4} \Longrightarrow$, we choose
$$c = .79 < \frac{1}{4} \times \left(\frac{4}{\sqrt{5}}\right)^2 = .8$$
- Thus the set:
$$\Omega_c = \{x \in R^2 | \ V(x) \ \leq \ .79\} \ \subset \ R_A.$$
- Estimating RoA by the set Ω_c is simple but conservative
- Alternatively Lasalle's theorem can be used. It provides an estimate of R_A

Example 14:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4(x_1 + x_2) - h(x_1 + x_2) \end{aligned}$$

where $h : R \longrightarrow R$ s.t. $h(0) = 0, \ \ \& xh(x) \ \geq \ 0 \ \ \forall |x| \leq 1$

- Consider the Lyap fcn candidate:

$$V(x) = x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x = 2x_1^2 + 2x_1x_2 + x_2^2$$

- Then $\dot{V} = -2x_1^2 - 6(x_1 + x_2)^2 - 2(x_1 + x_2)h(x_1 + x_2) \leq$ $-2x_1^2 - 6(x_1 + x_2)^2 = -x^T \begin{bmatrix} 8 & 6 \\ 6 & 6 \end{bmatrix} x, \ \ \forall |x_1 + x_2| \ \leq \ 1$
- $\therefore \dot{V}$ is n.d. in the set $G = \{x \in R^2 | \ |x_1 + x_2| \ \leq \ 1\}$.
- $(0, 0)$ is **a.s.**, to estimate R_A, first do it from Ω_c.

Example 14:

- Find the largest c s.t. $\Omega_c \subset G$. Now, c is given by

$$c = \min_{|x_1+x_2|=1} V(x) \text{ or}$$

$$c = \min\left\{ \min_{x_1+x_2=1} V(x), \quad \min_{x_1+x_2=-1} V(x) \right\}$$

- The first minimization yields

$$\begin{aligned} \min_{x_1+x_2=1} V(x) &= \min_{x_1} \left\{2x_1^2 + 2x_1(1-x_1) + (1-x_1)^2\right\} = 1 \quad \text{and} \\ \min_{x_1+x_2=-1} V(x) &= 1 \end{aligned}$$

- Hence, Ω_c with $c = 1$ is an estimate of R_A.
- A better (less conservative) estimate of R_A is possible.

Example 14:

- The key point is to observe that traj inside G cannot leave it through certain segment of the boundary $|x_1 + x_2| = 1$.
- Let $\quad \sigma = x_1 + x_2 \implies \partial G$ is given by $\sigma = 1$ and $\sigma = -1$
- We have
$$\begin{aligned} \frac{d}{dt}\sigma^2 &= 2\sigma(\dot{x}_1 + \dot{x}_2) = 2\sigma x_2 - 8\sigma^2 - 2\sigma h(\sigma) \\ &\leq 2\sigma x_2 - 8\sigma^2, \quad \forall |\sigma| \leq 1 \end{aligned}$$
- On the boundary $\sigma = 1 \implies \frac{d\sigma^2}{dt} \leq 2x_2 - 8 \leq 0 \quad \forall x_2 \leq 4$
- Hence, the traj on $\sigma = 1$ for which $x_2 \leq 4$ cannot move outside the set G since σ^2 is non-increasing
- Similarly, on the boundary $\sigma = -1$ we have
$$\frac{d\sigma^2}{dt} \leq -2x_2 - 8 \leq 0 \quad \forall x_2 \geq -4$$
- Hence, the traj on $\sigma = -1$ for which $x_2 \geq -4$ cannot move outside the set G.

Example 14:

- To define the boundary of G, we need to find two other segments to close the set.
- We can take them as the segments of Lyap. fcn surface
- Let c_1 be s.t. $V(x) = c_1$ intersects the boundary of $x_1 + x_2 = 1$ at $x_2 = 4$ and let c_2 be s.t. $V(x) = c_2$ intersects the boundary of $x_1 + x_2 = -1$ at $x_2 = -4$
- Then, we define $V(x) = \min c_1, c_2$, we have

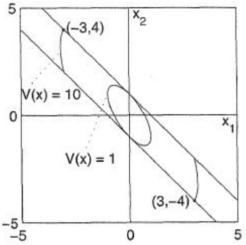
$$c_1 = V(x)\Big|_{\substack{x_1 = -3 \\ x_2 = 4}} = 10 \quad \& c_2 = V(x)\Big|_{\substack{x_1 = 3 \\ x_2 = -4}} = 10$$

Example 14:

- The set Ω is defined by

$$\Omega = \{x \in R^2 | V(x) \ \leq \ 10 \quad \& |x_1 + x_2| \ \leq \ 1\}$$

- This set is closed and bounded and **positively invariant**. Also, $\dot{V}$ is n.d. in Ω since $\Omega \ \subset \ G \ \Longrightarrow \ \Omega \ \subset \ R_A$.



Estimates of the region of attraction