Nonlinear Control
Lecture 4: Stability Analysis I

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Autonomous Systems
  Lyapunov Stability
  Variable Gradient Method
  Region of Attraction

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Linear System and Linearization

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Stability

Stability theory is divided into three parts:

1. Stability of equilibrium points
2. Stability of periodic orbits
3. Input/output stability

An equilibrium point (Equ. pt.) is:

- **Stable** if all solutions starting at nearby points stay nearby.
- **Asymptotically Stable** if all solutions starting at nearby points not only stay nearby, but also tend to the Equ. pt. as time approaches infinity.
- **Exponentially Stable**, if the rate of converging to the Equ. pt. is exponentially.

Lyapunov stability theorems give **sufficient conditions** for stability, asymptotic stability, and so on.

Lyapunov stability analysis can be used to show boundedness of the solution even when the system has no equilibrium points.

The theorems provide **necessary conditions** for stability are so-called converse theorems.
The most popular method for studying stability of nonlinear systems is introduced by a Russian mathematician named Alexander Mikhailovich Lyapunov.

Lyapunov’s work “The General Problem of Motion Stability” published in 1892 includes two methods:

- **Linearization Method**: studies nonlinear local stability around an Equ. point from stability properties of its linear approximation.
- **Direct Method**: not restricted to local motion. Stability of nonlinear system is studied by proposing a scalar energy-like function for the system and examining its time variation.

His work was then introduced by other scientists like Poincare and Lasalle.

Consider the autonomous system:

$$\dot{x} = f(x)$$

where $f : D \rightarrow \mathbb{R}^n$ is a locally Lip. function on a domain $D \subset \mathbb{R}^n$.

Let $\bar{x} \in D$ be an Equ. point, that is $f(\bar{x}) = 0$.

**Objective:** To characterize stability of $\bar{x}$.

**without loss of generality (wlog), let $\bar{x} = 0$**

- If $\bar{x} \neq 0$, introduce a coordinate transformation: $y = x - \bar{x}$, then

  $$\dot{y} = \dot{x} = f(y + \bar{x}) = g(y) \text{ with } g(0) = 0$$
The Equ. point $x = 0$ of $\dot{x} = f(x)$ is:

- **stable**, if for each $\epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ s.t.
  \[
  \|x(0)\| < \delta \implies \|x(t)\| < \epsilon \quad \forall t \geq 0
  \]

- **unstable**, if it is not stable

- **asymptotically stable**, if it is stable and $\delta$ can be chosen s.t.
  \[
  \|x(0)\| < \delta \implies \lim_{t \to \infty} x(t) = 0
  \]

∴ Lyapunov stability means that the system trajectory can be kept arbitrary close to the origin by starting sufficiently close to it.

An Equ. point which is Lyapunov stable but not asymptotically stable is called **Marginally stable**
Example: Van Der Pol Oscillator

- Van der Pol oscillator dynamics:
  \[
  \dot{x}_1 = x_2 \\
  \dot{x}_2 = -x_1 + (1 - x_1^2)x_2
  \]

- All system trajectories start except from origin, asymptotically approaches a limit cycle.
- \(.\) Even the system states remain around the Equ. point in a certain sense, they can not stay arbitrarily close to it.
- So the Equ. point is unstable.

Implicit in Lyapunov stability condition is that the sol. are defined \(\forall t \geq 0\).

This is not guaranteed by local Lip.

The additional condition imposed by Lyapunov theorem will ensure global existence of sol.
Lyapunov Stability

Physical Motivation

Consider the pendulum example (recall Lecture 2):

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2
\end{align*}
\]

- In first period it has two Equ. pts. \((x_1 = 0, x_2 = 0)\) \& \((x_1 = \pi, x_2 = 0)\)
- For frictionless pendulum, i.e. \(k = 0\) : trajectories are closed orbits in neighborhood of 1st Equ. pt. \(\to \epsilon - \delta\) requirement for stability is satisfied.
- However, it is not asymptotically stable.
- For Pendulum with friction, i.e. \(k > 0\)
  the 1st Equ. pt. is a stable focus \(\to \epsilon - \delta\) requirement for asymptotic stability is satisfied.
  the 2nd Equ. pt. is a saddle point \(\to \epsilon - \delta\) requirement is not satisfied \(\to\)
  it is unstable.
To generalize the phase-plane analysis, consider the energy associated with the pendulum:

\[ E(x) = \frac{1}{2}x^2 + \int_0^{x_1} \frac{g}{l} \sin y \, dy = \frac{1}{2}x^2 + \frac{g}{l}(1 - \cos x_1), \quad E(0) = 0 \]

- If \( k = 0 \), the system is conservative, i.e., there is no dissipation of energy:
  - \( E \) is constant during the motion of the system.
  - \( \dot{E} = 0 \) along the trajectory of the system.

- If \( k > 0 \), energy is being dissipated
  - \( \dot{E} < 0 \) along the trajectory of the system.
  - \( \therefore E \) starts to decrease until it eventually reaches zero, at that point \( x = 0 \).

- Lyapunov showed that certain other functions can be used instead of energy functions to determine stability of an equilibrium point.
Lyapunov’s Direct Method:

- Let $x = 0$ be an Equ. pt. for $\dot{x} = f(x)$. Let $V : D \rightarrow \mathbb{R}$, $D \subset \mathbb{R}^n$ be a continuously differentiable function on a neighborhood $D$ of $x = 0$, s.t.
  1. $V(0) = 0$
  2. $V(x) > 0$ in $D \setminus \{0\}$
  3. $\dot{V}(x) \leq 0$ in $D$

Then $x = 0$ is stable.

Moreover, if $\dot{V}(x) < 0$ in $D \setminus \{0\}$ then $x = 0$ is asymptotically stable.

- The continuously differentiable function $V(x)$ is called a Lyapunov function.

- The surface $V(x) = c$, for some $c > 0$ is called a Lyapunov surface or level surface.
Lyapunov Stability

► when $\dot{V} \leq 0 \implies$
when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\}$ and traps inside $\Omega_c$.

► when $\dot{V} < 0 \implies$
trajectories move from one level surface to an inner level with smaller $c$ till $V(x) = c$ shrinks to zero as time goes on.
Lyapunov Stability

- A function satisfying $V(0) = 0$ & $V(x) > 0$ in $D - \{0\}$ is said to be Positive Definite (p.d.)

- If it satisfies a weaker condition $V(x) \geq 0$ for $x \neq 0$ is said to be Positive Semi-Definite (p.s.d.)

- A function is Negative Definite (n.d.) or Negative Semi-Definite (n.s.d.) if $-V(x)$ is p.d. or p.s.d., respectively.

- Lyapunov theorem states that:
  The origin is stable if there is a continuously differentiable, p.d. function $V(x)$ s.t. $\dot{V}(x)$ is n.s.d., and is asymptotically if $\dot{V}(x)$ is n.d.

- Note that when $x$ is a vector:

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i = \left[ \frac{\partial V}{\partial x_1} \cdots \frac{\partial V}{\partial x_n} \right] \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$
Lyapunov Stability

- A class of scalar functions for which sign definition can be easily checked is "quadratic functions:"

\[ V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j P_{ij} \]

where \( P = P^T \) is a real matrix.

- \( V(x) \) is p.d./p.s.d. iff \( \lambda_i(P) > 0 \) or \( \lambda_i(P) \geq 0 \), \( i = 1 \ldots n \)

- \( \lambda_i(P) > 0 \) or \( \lambda_i(P) \geq 0 \), \( i = 1 \ldots n \) iff all leading principle minors of \( P \) are positive or non-negative, respectively.

- If \( V(x) \) is p.d. (p.s.d.), we say the matrix \( P \) is p.d. (p.s.d.) and write \( P > 0 \) (\( P \geq 0 \)).
Example 1

\[ V(x) = ax_1^2 + 2x_1x_3 + ax_2^2 + 4x_2x_3 + ax_3^2 \]

\[ = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T Px \]

- The leading principle minors are
  \[ \det(a) = a; \quad \det \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a^2; \quad \det \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} = a(a^2 - 5) \]

\[ \therefore V(x) \text{ is p.s.d. if } a \geq \sqrt{5}, \quad V(x) \text{ is p.d. if } a > \sqrt{5} \]

- For n.d. the leading principle minors of
  - \( -P \) should be positive. OR
  - \( P \) should alternate in sign with the first one neg. (odds: neg., even: pos.)

\[ \therefore V(x) \text{ is n.s.d. if } a \leq -\sqrt{5}, \quad V(x) \text{ is n.d. if } a < -\sqrt{5}, \]
\[ V(x) \text{ is sign indefinite for } -\sqrt{5} < a < \sqrt{5} \]
Example 2

Consider \( \dot{x} = -g(x) \) where \( g(x) \) is locally Lip. on \((-a, a) \) & \( g(0) = 0 \), \( xg(x) > 0 \), \( \forall x \neq 0 \), \( x \in (-a, a) \). stability?

 origin is Equ. pt.

 Solution 1:

 starting on either side of the origin will have to move toward the origin due to the sign of \( \dot{x} \)

 \( \therefore \) Origin is an isolated Equ. pt. and is asymptotically stable.

 Solution 2: using Lyapunov theorem:

 Consider the function \( V(x) = \int_0^x g(y)dy \) over \( D = (-a, a) \).

 \( V(x) \) is continuously differentiable, \( V(0) = 0 \) and \( V(x) > 0 \), \( \forall x \neq 0 \). \( V \) is a valid Lyapunov candidate

 To see if it is really a Lyap. fcn, we have to take its derivative along system trajectory: \( \dot{V}(x) = \frac{\partial V}{\partial x}(-g(x)) = -g^2(x) < 0 \), \( \forall x \in D - \{0\} \)

 \( \therefore \) \( V(x) \) is a valid Lyap. fcn \( \rightsquigarrow \) the origin is asymptotically stable.
Example 3: Frictionless Pendulum

\[
\begin{align*}
\dot{x}_1 & = x_2 \\
\dot{x}_2 & = -\frac{g}{l} \sin x_1
\end{align*}
\]

- Study stability of the Equ. pt. at the origin.
- A natural Lyap. fcn is the energy fcn:

\[
V(x) = \frac{g}{l} (1 - \cos x_1) + \frac{1}{2} x_2^2
\]

- \(V(0) = 0\) and \(V(x)\) is p.d. over the domain \(-2\pi \leq x_1 \leq 2\pi\).
- \[\dot{V} = \frac{g}{l} \dot{x}_1 \sin x_1 + x_2 \dot{x}_2 = \frac{g}{l} x_2 \sin x_1 - \frac{g}{l} x_2 \sin x_1 = 0\]
- \(V(x)\) satisfies the condition of the Lyap. Theorem \(\leadsto\) origin is **stable**
Example 4: Pendulum with Friction

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2
\end{align*}
\]

- Take the energy fcn as a Lyapunov fcn candidate

\[V(x) = \frac{g}{l} (1 - \cos x_1) + \frac{1}{2} x_2^2\]

- \[\dot{V} = -\frac{k}{m} x_2^2\]

- \[\dot{V}(x)\] is n.s.d. It is not since n.d. since \(\dot{V} = 0\) for \(x_2 = 0\) and all \(x_1 \neq 0\). the origin is only stable.

- But, phase portrait showed asymptotic stability!!

- Toward this end, let’s choose:

\[V(x) = \frac{1}{2} x^T P x + \frac{g}{l} (1 - \cos x_1)\]
where $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ is p.d. ($P_{11} > 0$, $P_{22} > 0$, $P_{11}P_{22} - P_{12}^2 > 0$)

\[
\dot{V} = \frac{1}{2}(\dot{x}^T P \dot{x} + x^T P \ddot{x}) + \frac{g}{l} \dot{x}_1 \sin x_1 = \frac{g}{l} (1 - P_{22}) x_2 \sin x_1 \\
- \frac{g}{l} P_{12} x_1 \sin x_1 + (P_{11} - P_{12} \frac{k}{m}) x_1 x_2 + (P_{12} - P_{22} \frac{k}{m}) x_2^2
\]

- Select $P$ s.t. $\dot{V}$ is n.d. (cancel sign indefinite factors: $x_2 \sin x_1$ and $x_1 x_2$)
- $P_{22} = 1$, $P_{11} = \frac{k}{m} P_{12}$, $0 < P_{12} < \frac{k}{m}$ (for $V(x)$ to be p.d., take $P_{12} = \frac{1}{2} \frac{k}{m}$)
- $\dot{V} = -\frac{1}{2} \frac{g}{l} \frac{k}{m} x_1 \sin x_1 - \frac{1}{2} \frac{k}{m} x_2^2$

- $x_1 \sin x_1 > 0 \ \forall \ 0 < |x_1| < \pi$, defining a domain $D$ by $D = \{ x \in \mathbb{R}^2 | \ |x_1| < \pi \}$
- $V(x)$ is p.d. and $\dot{V}$ is n.d. over $D$. Thus, origin is asymptotically stable (a.s.) by the theorem.
How Search for A Lyapunov Function?

- Lyapunov theorem is only **sufficient**.
- Failure of a Lyap. fcn candidate to satisfy the theorem **does not** mean the Equ. pt. is unstable.

**Variable Gradient Method**

- Idea is working backward:
  - Investigated an expression for $\dot{V}(x)$ and go back to choose the parameters of $V(x)$ so as to make $\dot{V}(x)$ n.d. (n.s.d)
- Let $V = V(x)$ and $g(x) = \nabla_x V = \left(\frac{\partial V}{\partial x}\right)^T$
- Then $\dot{V} = \frac{\partial V}{\partial x} f = g^T f$
- Choose $g(x)$ s.t. it would be the gradient of a p.d. fcn $V$ and make $\dot{V}$ n.d. (n.s.d)
- $g(x)$ is the gradient of a scalar fcn iff the Jacobian matrix $\frac{\partial g}{\partial x}$ is symmetric:

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall i, j = 1, \ldots, n$$
Variable Gradient Method

- Select \( g(x) \) s.t. \( g^T(x)f(x) \) is n.d.

- Then, \( V(x) \) is computed from the integral:

\[
V(x) = \int_{0}^{x} g(y)dy = \int_{0}^{x} \sum_{i=1}^{n} g_i(y)dy_i
\]

- The integration is taken over any path joining the origin to \( x \). This can be done along the axes:

\[
V(x) = \int_{0}^{x_1} g_1(y_1, 0, ..., 0)dy_1 + \int_{0}^{x_2} g_2(x_1, y_2, 0, ..., 0)dy_2 + \ldots + \int_{0}^{x_n} g_n(x_1, x_2, ..., y_n)dy_n
\]

- By leaving some parameters of \( g \) undetermined, one would try to choose them so that \( V \) is p.d.
Example 5:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -h(x_1) - ax_2
\end{align*}
\]

where \( a > 0 \), \( h(.) \) is locally Lip.,
\( h(0) = 0 \), \( yh(y) > 0 \), \( \forall \ y \neq 0 \), \( y \in (-b, c) \), \( b, c > 0 \).

▶ The pendulum is a special case of this system.
▶ Find proper Lyapunov function?
▶ Applying variable gradient method, we must find \( g(x) \) s.t. \( \frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} \)
▶ \( \dot{V}(x) = g_1(x)x_2 - g_2(x)(h(x_1) + ax_2) < 0 \), \( \forall x \neq 0 \) and
\[
V(x) = \int_0^x g^T(y)dy > 0 \text{ for } x \neq 0
\]
▶ Choose \( g(x) = \begin{bmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2 \end{bmatrix} \) where \( \alpha, \beta, \gamma, \delta \) to be determined
To satisfy the symmetry req., we need

\[ \beta(x) + \frac{\partial \alpha}{\partial x_2} x_1 + \frac{\partial \beta}{\partial x_2} x_2 = \gamma(x) + \frac{\partial \gamma}{\partial x_1} x_1 + \frac{\partial \delta}{\partial x_1} x_2 \]

\[ \dot{V}(x) = \alpha(x) x_1 x_2 + \beta(x) x_2^2 - a \gamma(x) x_1 x_2 - a \delta(x) x_2^2 - \delta(x) x_2 h(x_1) - \gamma(x) x_1 h(x_1) \]

To cancel the cross terms, let
\[ \alpha(x) x_1 - a \gamma(x) x_1 - \delta(x) h(x_1) = 0 \]

\[ \therefore \dot{V}(x) = -(a \delta(x) - \beta(x)) x_2^2 - \gamma(x) x_1 h(x_1) \]

For simplification, let \( \delta(x) = \delta = cte, \gamma(x) = \gamma = cte, \beta(x) = \beta = cte \)

\[ \therefore \alpha(x) \text{ only depends on } x_1 \]

\[ \text{symmetry is satisfied if } \beta = \gamma. \]

\[ g(x) = \begin{bmatrix} a \gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix} \]
By integration, we get

\[
V(x) = \int_0^{x_1} \left( a\gamma y_1 + \delta h(y_1) \right) dy_1 + \int_0^{x_2} \left( \gamma x_1 + \delta y_2 \right) dy_2 \\
= \frac{1}{2} a\gamma x_1^2 + \delta \int_0^{x_1} h(y) dy + \gamma x_1 x_2 + \frac{1}{2} \delta x_2^2 \\
= \frac{1}{2} x^T P x + \delta \int_0^{x_1} h(y) dy
\]

where \( P = \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix} \).

Choosing \( \delta > 0, \ 0 < \gamma < a\delta \implies V \) is p.d. \& \( \dot{V} \) is n.d.

e.g., taking \( \gamma = a k \delta, \ 0 < k < 1 \) yields

\[
V(x) = \frac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y) dy
\]

over \( D = \{ x \in R^n \mid -b < x_1 < c \} \) conditions of the theorem are satisfied

\( \implies x = 0 \) is asymptotically stable.
Region of Attraction

- For asymptotically stable Equ. pt.: How far from the origin can the trajectory be and still converges to the origin as $t \rightarrow \infty$?
- Let $\phi(t, x)$ be the sol. of $\dot{x} = f(x)$ starting at $x_0$. Then, the Region of Attraction (RoA) is defined as the set of all pts. $x$ s.t. $\lim_{t \rightarrow \infty} \phi(t, x) = 0$
- Lyap. fcn can be used to estimate the RoA:
  - If there is a Lyap. fcn. satisfying asymptotic stability over domain $D$, and set $\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\}$ is bounded and contained in $D$
  - $\therefore$ all trajectories starting in $\Omega_c$ remains there and converges to 0 at $t \rightarrow \infty$
- Under what condition the RoA be $\mathbb{R}^n$ (i.e., the Equ. pt. is globally asymptotically stable (g.a.s))?
  - the conditions of stability theory must hold globally, i.e. $D = \mathbb{R}^n$
  - This not enough!
  - for large $c$, the set $\Omega_c$ should be kept bounded.
    i.e., reduction of $V(x)$ should also result in reduction of $\|x\|$. 
Region of Attraction

Example: \( V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2 \)

- It’s clear that \( V(x) \) can get smaller, but \( x \) grows unboundedly.

Babashin-Krasovskii Theorem: Let \( x = 0 \) be an Equ. pt. of \( \dot{x} = f(x) \).

Let \( V : R^n \rightarrow R \) be a continuously differentiable fcn. s.t.:

- \( V(0) = 0 \)
- \( V(x) > 0, \forall x \neq 0 \)
- \( \|x\| \rightarrow \infty \implies V(x) \rightarrow \infty \) (i.e. it is radially unbounded)
- \( \dot{V} < 0, \forall x \neq 0 \)

then \( x = 0 \) is globally asymptotically stable.
Example 6: Globally Asymptotically Stable

- Reconsider Example 5 \( h(0) = 0, \ x_h(x) > 0, \ \forall x \neq 0, \ x \in (-a, a) \)
- but assume that \( x_h(x) > 0 \) hold for all \( x \neq 0 \).
  - The Lyap. fcn:
    \[
    V(x) = \frac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)dy
    \]
    - is p.d. \( \forall x \in \mathbb{R}^2 \)
    - \( V(x) \) is radially unbounded.
    - \( \dot{V} = -a\delta(1 - k)x_2^2 - ak\delta x_1 h(x_1) < 0, \ \forall x \in \mathbb{R}^2 \)
    - \( \therefore \) origin is g.a.s.

- Important: Since the origin is g.a.s., then it must be the unique Equ. pt. of the system
- g.a.s. is not satisfied for multiple Equ. pt. problem such as pendulum.
**Instability Theorem**

**Chetaev’s Theorem** Let $x = 0$ be an Equ. pt. of $\dot{x} = f(x)$. Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable fcn such that $V(0) = 0$ and $V(x_0) > 0$ for some $x_0$ with arbitrary small $\|x_0\|$. Define a set $\nu = \{x \in B_r | V(x) > 0\}$ where $B_r = \{x \in \mathbb{R}^n \|x\| < r\}$ and suppose that $\dot{V}(x)$ is p.d. in $\nu$. Then, $x = 0$ is unstable.

**Example:**

\[
\begin{align*}
\dot{x}_1 &= x_1 + g_1(x) \\
\dot{x}_2 &= -x_2 + g_2(x)
\end{align*}
\]

where $|g_i(x)| \leq k\|x\|^2_2$ in a neighborhood $D$ of origin.

The inequality implies, $g_i(0) = 0 \implies$ origin is an Equ. pt.

**Consider:** $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$

- On the line $x_2 = 0$, $V(x) > 0$.
- $\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$

- Since $|x_1 g_1(x) - x_2 g_2(x)| \leq \sum_{i=1}^{2} |x_i| |g_i(x)| \leq 2k\|x\|^3_2$

- $\therefore \dot{V}(x) \geq \|x\|^2_2 - 2k\|x\|^3_2 = \|x\|^2_2(1 - 2k\|x\|_2)$

- Choosing $r$ s.t. $B_r \subset D$ and $r < \frac{1}{2}k \implies$ origin is unstable.
Invariance Principle

- Recall the pendulum example:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2
\end{align*}
\]

\[\dot{V}(x) = -\frac{k}{m} x_2^2\] which is n.s.d.

- Lyap. theorem shows **only stability**. However,
  - \(\dot{V}\) is negative everywhere except at \(x_2 = 0\) where \(\dot{V} = 0\).
  - To get \(\dot{V} = 0\), the trajectory must be confined to \(x_2 = 0\)
  - Now, from the model \(x_2(t) \equiv 0 \implies \dot{x}_1 \equiv 0 \implies x_1(t) = \text{cte}\) and \(x_2(t) \equiv 0 \implies \dot{x}_2 \equiv 0 \implies \sin x_1 \equiv 0\)
  - Hence, on the segment \(-\pi < x_1 < \pi\) of \(x_2 = 0\) line, the system can maintain \(\dot{V} = 0\) only at \(x = 0\).

- Therefore, \(V(x)\) decrease to zero and \(x(t) \rightarrow 0\) as \(t \rightarrow \infty\).

- The idea follows from **LaSalle’s Invariance Principle**
Invariance Principle

- A set $M$ is said to be a **positively invariant set** with respect to $\dot{x} = f(x)$, if $x(0) \in M \Rightarrow x(t) \in M$, $\forall t \geq 0$.
  - If a solution belongs to $M$ at some time instant, then it belongs to $M$ for all future time.
- $\therefore$ Equ. points and limit cycle are invariant sets.
- Also the set $\Omega = \{x \in R^n | V(x) \leq c\}$ with $\dot{V} \leq 0$, $\forall x \in \Omega$ is a positively invariant set.
Lasalle’s Theorem:

- Let $\Omega$ be a compact set with property that every solution of $\dot{x} = f(x)$ starting in $\Omega$ remains in $\Omega$ for all future time.
  - Let $V : \Omega \to \mathbb{R}$ be a continuously differentiable fcn s.t. $\dot{V}(x) \leq 0$ in $\Omega$.
  - Let $E$ be the set of all pts in $\Omega$ where $\dot{V}(x) = 0$
  - Let $M$ be the largest invariant set in $E$.

Then, every sol. starting in $\Omega$ approaches $M$ as $t \to \infty$

- Unlike Lyap. theorem, Lasalle’s theorem **does not** require $V(x)$ to be p.d.
- Only $\Omega$ should be bounded
  - If $V$ is p.d. $\Rightarrow \Omega = \{ x \in \mathbb{R}^n | V(x) \leq c \}$ is bounded for suff. small $c$
  - If $V$ is radially unbounded $\Rightarrow \Omega$ is bounded for all $c$ no matter $V$ is p.d. or not

- To show a.s. of the origin $\to$ show largest invariant set in $E$ is the origin.
- $\therefore$ Show that no solution can stay forever in $E$ other than $x = 0$. 
Lasalle’s Theorem:
Barbashin and Krasovskii Corollaries

Corollary 1: Let $x = 0$ be an Equ. pt of $\dot{x} = f(x)$. Let $V : D \rightarrow R$ be a continuously differentiable p.d. fcn on a domain $D$ containing the origin $x = 0$, s.t. $\dot{V}(x) \leq 0$ in $D$. Let $S = \{x \in D|\dot{V} = 0\}$ and suppose that no solution can stay identically in $S$, other than the trivial solution $x(t) = 0$. Then, the origin is a.s.

Corollary 2: Let $x = 0$ be an Equ. pt. of $\dot{x} = f(x)$. Let $V : R^n \rightarrow R$ be a continuously differentiable, radially unbounded, p.d. fcn s.t. $\dot{V}(x) \leq 0 \ \forall x \in R^n$. Let $S = \{x \in R^n|\dot{V} = 0\}$ and suppose that no solution can stay in $S$ forever except $x = 0$. Then, the origin is g.a.s.
Example 6:

- Consider

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -g(x_1) - h(x_2)
\end{align*}
\]

where \( g(.) \) & \( h(.) \) are locally Lip. and satisfy
\[
\begin{align*}
g(0) &= 0, \quad yg(y) > 0 \quad \forall y \neq 0, \quad y \in (-a, a) \\
h(0) &= 0, \quad yh(y) > 0 \quad \forall y \neq 0, \quad y \in (-a, a)
\end{align*}
\]

- The system has an isolated Equ. pt. at origin. Let

\[
V(x) = \frac{1}{2}x_2^2 + \int_{0}^{x_1} g(y)dy
\]

\[
D = \{x \in \mathbb{R}^2| -a < x_i < a\} \implies V(x) > 0 \text{ in } D
\]

\[
\dot{V} = g(x_1)x_2 + x_2(-g(x_1) - h(x_2)) = -x_2h(x_2) \leq 0
\]

- Thus, \( V \) is n.s.d. and the origin is stable by Lyap. theorem
Example 6:

- Using Lasalle’s theorem, define $S = \{ x \in D | \dot{V} = 0 \}$
  - $\dot{V} = 0 \implies x_2 h(x_2) = 0 \implies x_2 = 0$, since $-a < x_2 < a$
  - Hence $S = \{ x \in D | x_2 = 0 \}$. Suppose $x(t)$ is a traj. $\in S \ \forall t$
  - $\therefore x_2(t) \equiv 0 \implies \dot{x}_1 \equiv 0 \implies x_1(t) = c$, where $c \in (-a, a)$. Also
    $x_2(t) \equiv 0 \implies \dot{x}_2 \equiv 0 \implies g(c) = 0 \implies c = 0$
  - $\therefore$ Only solution that can stay in $S \ \forall t \geq 0$ is the origin $\implies x = 0$ is a.s.

- Now, Let $a = \infty$ and assume $g$ satisfy:
  $$\int_{0}^{y} g(z)dz \longrightarrow \infty \text{ as } |y| \longrightarrow \infty.$$  
  - The Lyap. fcn $V(x) = \frac{1}{2}x_2^2 + \int_{0}^{x_1} g(y)dy$ is radially unbounded.
  - $\dot{V} \leq 0$ in $R^2$ and note that $S = \{ x \in R^2 | \dot{V} = 0 \} = \{ x \in R^2 | x_2 = 0 \}$ contains no solution other than origin $\implies x = 0$ is g.a.s.
Summary

▶ **Lyapunov Direct Method:** The origin of an autonomous system $\dot{x} = f(x)$ is stable if there is a continuously differentiable, p.d. function $V(x)$ s.t. $\dot{V}(x)$ is n.s.d., and it is asymptotically if $\dot{V}(x)$ is n.d.

▶ $V(x)$ is p.d./p.s.d. iff $\lambda_i\{P\} > 0/\lambda_i\{P\} \geq 0$ iff all leading principle minors of $P$ are positive / non-negative.

▶ **Variable Gradient Method:** To find a Lyap fcn: Choose $g(x)$ s.t.

1. $\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}$
2. $g^T(x)f(x)$ is n.d. (n.s.d)
3. $V(x) = \int_0^{x_1} g_1(y_1, 0, ..., 0)dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, ..., 0)dy_2 + ... + \int_0^{x_n} g_n(x_1, x_2, ..., y_n)dy_n$ is P.d.

▶ **Babashin-Krasovskii Theorem:** Let $x = 0$ be an Equ. pt. of $\dot{x} = f(x)$.

Let $V : R^n \rightarrow R$ be a continuously differentiable fcn. s.t.: $V(0) = 0$, $V(x) > 0$, $\forall x \neq 0$, $\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$ (i.e. it is radially unbounded), $\dot{V} < 0$, $\forall x \neq 0$ then $x = 0$ is globally asymptotically stable
Another method for study a.s is defined based on Lasalle Theorem:

**Corollary 1:** Let $x = 0$ be an Equ. pt of $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a continuously differentiable p.d. fcn on a domain $D$ containing the origin $x = 0$, s.t. $\dot{V}(x) \leq 0$ in $D$. Let $S = \{x \in D | \dot{V} = 0\}$ and suppose that no solution can stay identically in $S$, other than the trivial solution $x(t) = 0$. Then, the origin is **a.s.**

- The origin is **g.a.s.** if $D = \mathbb{R}^n$, and $V(x)$ is radially unbounded.
Invariance Principle

Lasalle’s theorem can also extend the Lyap. theorem in three different directions:

1. It gives an estimate of the RoA not necessarily in the form of \( \Omega_c = \{ x \in R^n \mid V(x) \leq c \} \). The set can be any positively invariant set which leads to less conservative estimate.


3. The function \( V(x) \) does not have to be positive definite.

Example 7: shows how to use Lasalle’s theorem for system with Equ. sets rather than isolated Equ. pts.

A simple adaptive control problem:

\[
\dot{x} = ax + u \quad \text{a unknown}
\]

with the adaptive control law

\[
u = -kx; \quad \dot{k} = \gamma x^2, \quad \gamma > 0
\]
Example 7:

Let $x_1 = x, x_2 = k$, we get:

\[
\begin{align*}
\dot{x}_1 &= -(x_2 - a)x_1 \\
\dot{x}_2 &= \gamma x_1^2
\end{align*}
\]

- The line $x_1 = 0$ is an Equ. set
- Show that the traj. of closed-loop system approaches this set as $t \to \infty$
  - i.e. the adaptive system regulates $y$ to zero ($x_1 \to 0$ as $t \to \infty$).
- Consider the Lyap. fcn candidate:

\[
V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2
\]

where $b > a$.
- $\dot{V} = -x_1^2(b - a) \leq 0$
- The set $\Omega = \{x \in \mathbb{R}^n | V(x) \leq c\}$ is a compact positively invariant set.
Example 7:
- \( V(x) \) is radially unbounded \( \implies \) Lasalle’s theorem conditions are satisfied with the set \( E \) as \( E = \{ x \in \Omega | x_1 = 0 \} \)
- Since any pt on \( x_1 = 0 \) line is an Equ. pt, \( E \) is an invariant set: \( M = E \).
- Hence, every trajectory starting in \( \Omega \) approaches \( E \) as \( t \to \infty \), i.e. \( x_1(t) \to 0 \) as \( t \to \infty \).
- \( V \) is radially unbounded \( \implies \) the result is global
- Note that in the above example the Lyapunov function depends on a constant \( b \) which is required to satisfy \( b > a \)
- But it is not known \( \iff \) we may not know the constant \( b \) explicitly, but we know that it always exists.
- This highlights another feature of Lyapunov’s method:
  - In some situations, we may be able to assert the existence of a Lyapunov function that satisfies the conditions, even though we may not explicitly know that function.
Linear System and Linearization

- Given $\dot{x} = Ax$, the Equ. pt. is at origin
- It is isolated iff $\det A \neq 0$,
- System has an Equ. subspace if $\det A = 0$, the subspace is the null space of $A$.
- The linear system cannot have multiple isolated Equ. pt. since
  - Linearity requires that if $x_1$ and $x_2$ are Equ. pts., then all pts. on the line connecting them should also be Equ. pts.

- **Theorem:** The Equ. pt. $x = 0$ of $\dot{x} = Ax$ is stable iff all eigenvalues of $A$ satisfy $\Re\{\lambda_i\} \leq 0$ and every eigenvalue with $\Re\{\lambda_i\} = 0$ and algebric multiplicity $q_i \geq 2$, $\text{rank}(A - \lambda_i I) = n - q_i$, where $n$ is dimension of $x$.
  The Equ. pt. $x = 0$ is globally asymptotically stable iff $\Re\lambda_i < 0$. 
Linear Systems and Linearization

- When all eigenvalues of $A$ satisfy $\text{Re}\lambda_i < 0$, $A$ is called a **Hurwitz matrix**.
- Asymptotic stability can be verified by using Lyapunov’s method:
  - Consider a quadratic Lyapunov function candidate:
    \[
    V(x) = x^T P x, \quad P = P^T > 0
    \]
    \[
    \dot{V} = \dot{x}^T Px + x^T \dot{P} x, \quad x^T (A^T P + PA)x \triangleq -x^T Q x
    \]

  where

  \[
  A^T P + P A = -Q, \quad Q = Q^T \quad \text{Lyapunov Equation}
  \]

- If $Q$ is p.d., then we conclude that $x = 0$ is **g.a.s.**
- We can proceed alternatively as follows:
  - Start by choosing $Q = Q^T$, $Q > 0$, then solve the Lyapunov eqn. for $P$.
  - If $P > 0$, then $x = 0$ is **g.a.s.**
Linear Systems and Linearization

**Theorem:** A matrix $A$ is a stable matrix, i.e. $\text{Re} \, \lambda_i < 0$ iff for every given $Q = Q^T > 0$, $\exists \ P = P^T > 0$ that satisfies the Lyap. eq. Moreover, if $A$ is a stable matrix, then $P$ is unique.

**Example 8:**

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

Let $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Q^T > 0$

denote $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = P^T > 0$

The Lyap. eq. $A^T P + P A = -Q$ becomes

$$2 \ P_{12} = -1$$
$$-P_{11} - P_{12} + P_{22} = 0$$
$$-2 \ P_{12} - 2P_{22} = -1$$
Linear Systems and Linearization

\[
\begin{bmatrix}
0 & 2 & 0 \\
-1 & -1 & 1 \\
0 & -2 & -2 \\
\end{bmatrix}
\begin{bmatrix}
P_{11} \\
P_{12} \\
P_{22} \\
\end{bmatrix}
= 
\begin{bmatrix}
-1 \\
0 \\
-1 \\
\end{bmatrix}
\implies
\begin{bmatrix}
P_{11} \\
P_{12} \\
P_{22} \\
\end{bmatrix}
= 
\begin{bmatrix}
1.5 \\
-0.5 \\
1 \\
\end{bmatrix}
\]  

\(1\)

Let \(P = P^T = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix} > 0 \implies x = 0 \) is g.a.s

Remark: Computationally, there is no advantages in computing the eigenvalues of \(A\) over solving Lyap. eqn.
Consider $\dot{x} = f(x)$ where $f : D \rightarrow \mathbb{R}^n$, $D \subset \mathbb{R}^n$, is continuously differentiable. Let $x = 0$ is in the interior of $D$ and $f(0) = 0$.

In a small neighborhood of $x = 0$, the nonlinear system $\dot{x} = f(x)$ can be linearized by $\dot{x} = Ax$.

Proof: Ref. Khalil's book

**Theorem (Lyapunov's First Method):**

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ where $f : D \rightarrow \mathbb{R}^n$ is continuously differentiable and $D$ is a neighborhood of origin. Let $A = \frac{\partial f}{\partial x} \bigg|_{x=0}$, then

1. $x = 0$ is *asymptotically stable* if $\text{Re} \lambda_i < 0$, $i = 1, \ldots, n$
2. $x = 0$ is *unstable* if $\text{Re} \lambda_i > 0$, for one or more eigenvalues
Linear Systems and Linearization

Example 9: \( \dot{x} = ax^3 \)

- Linearization about \( x = 0 \) yields:

\[
A = \left. \frac{\partial f}{\partial x} \right|_{x=0} = 3ax^2 \bigg|_{x=0} = 0
\]

- Linearization fails to determine stability

- If \( a < 0 \), \( x = 0 \) is a.s.

- To see this, let \( V(x) = x^4 \implies \dot{V} = 4x^3 \dot{x} = 4ax^6 \)

- If \( a > 0 \), \( x = 0 \) is unstable

- If \( a \leq 0 \), \( x = 0 \) is stable, starting at any \( x \), remains in \( x \)

Example 10:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\left(\frac{g}{l}\right) \sin x_1 - \left(\frac{k}{m}x_2\right)
\end{align*}
\]
Linear Systems and Linearization

▶ Linearization about 2 Equ. pts. $(0,0)$ & $(\pi,0)$:

\[
A = \frac{\partial f}{\partial x} = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\frac{g}{l} \cos x_1 & -\frac{k}{m}
\end{bmatrix}
\]

▶ At $(0,0)$

- \[ A = \begin{bmatrix}
0 & 1 \\
-\frac{g}{l} & -\frac{k}{m}
\end{bmatrix} , \quad \Rightarrow \lambda_{1,2} = -\frac{1}{2} \frac{k}{m} \pm \frac{1}{2} \sqrt{\left(\frac{k}{m}\right)^2 - 4 \frac{g}{l}} \]
- \[ \therefore \forall g, k, l, m > 0 \implies \Re(\lambda_1, \lambda_2) < 0 \implies \text{x} = 0 \text{ is a.s.} \]
- \[ \text{If } k = 0 \implies \Re(\lambda_1, \lambda_2) = 0 \implies \text{eigenvalues on } j\omega \text{ axis, rank}(A - \lambda_i I) \neq 0 \]
- \[ \therefore \text{stability cannot be determined.} \]

▶ At $(\pi,0)$

- \[ A = \begin{bmatrix}
0 & 1 \\
\frac{g}{l} & -\frac{k}{m}
\end{bmatrix} , \quad \Rightarrow \lambda_{1,2} = -\frac{1}{2} \frac{k}{m} \pm \frac{1}{2} \sqrt{\left(\frac{k}{m}\right)^2 + 4 \frac{g}{l}} \]
- \[ \therefore \forall g, k, l, m > 0 \implies \text{there is one eigenvalue in the open right-half plane} \implies \text{x} = 0 \text{ is unstable.} \]
Example: Robot Manipulator

- **Dynamics:**

\[ M(q)\ddot{q} + C(q, \dot{q})\dot{q} + B\dot{q} + g(q) = u \]  

where \( M(q) \) is the \( n \times n \) inertia matrix of the manipulator

- \( C(q, \dot{q})\dot{q} \) is the vector of Coriolis and centrifugal forces

- \( g(q) \) is the term due to the Gravity

- \( B\dot{q} \) is the viscous damping term

- \( u \) is the input torque, usually provided by a DC motor

- **Objective:** To regulate the joint position \( q \) around desired position \( q_d \).

- A common control strategy PD+Gravity:

\[ u = K_P\ddot{q} - K_D\dot{q} + g(q) \]

where \( \ddot{q} = q_d - q \) is the error between the desired and actual position

- \( K_P \) and \( K_D \) are diagonal positive proportional and derivative gains.
Example: Robot Manipulator

Consider the following Lyapunov function candidate:

\[ V = \frac{1}{2} \dot{q}^T M(q) \dot{q} + \frac{1}{2} \ddot{q}^T K_P \ddot{q} \]

- The first term is the kinetic energy of the robot and the second term accounts for “artificial potential energy” associated with virtual spring in PD control law (proportional feedback \( K_p \ddot{q} \))

- Physical properties of a robot manipulator:
  1. The inertia matrix \( M(q) \) is positive definite
  2. The matrix \( \dot{M}(q) - 2C(q, \dot{q}) \) is skew symmetric

- \( V \) is positive in \( \mathbb{R}^n \) except at the goal position \( q = q^d \), \( \dot{q} = 0 \)

\[ \dot{V} = \dot{q}^T M(q) \ddot{q} + \frac{1}{2} \dot{q}^T \dot{M}(q) \dot{q} - \dot{q}^T K_P \ddot{q} \]

- Substituting \( M(q) \ddot{q} \) from (2) into the above equation yields

\[ \dot{V} = \dot{q}^T (u - C(q, \dot{q}) \dot{q} - B \dot{q} - g(q)) + \frac{1}{2} \dot{q}^T \dot{M}(q) \dot{q} - \dot{q}^T K_P \ddot{q} \]

\[ = \dot{q}^T (u - B \dot{q} - K_P \ddot{q} - g(q)) + \frac{1}{2} \dot{q}^T (\dot{M}(q) - 2C(q, \dot{q})) \dot{q} \]
Example: Robot Manipulator

\[ \dot{V} = \dot{\mathbf{q}}^T (u - B\dot{\mathbf{q}} - K_P\ddot{\mathbf{q}} - g(\mathbf{q})) \]

- where \( \dot{M} - 2C \) is skew symmetric \( \Rightarrow \dot{\mathbf{q}}^T (\dot{M}(\mathbf{q})\dot{\mathbf{q}} - C(\mathbf{q}, \dot{\mathbf{q}}))\dot{\mathbf{q}} = 0 \)
- Substitute PD control law for \( u \), we get:
  \[ \dot{V} = -\dot{\mathbf{q}}^T (K_D + B)\dot{\mathbf{q}} \leq 0 \] (3)

- The goal position is stable since \( V \) is non-increasing
- Use the invariant set theorem:
  - Suppose \( \dot{V} \equiv 0 \), then (3) implies that \( \dot{\mathbf{q}} \equiv 0 \) and hence \( \ddot{\mathbf{q}} \equiv 0 \)
  - From Equ. of motion (2) with PD control, we have
    \[ M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + B\dot{\mathbf{q}} = K_P\ddot{\mathbf{q}} - K_D\dot{\mathbf{q}} \]
    we must then have \( 0 = K_P\ddot{\mathbf{q}} \) which implies that \( \ddot{\mathbf{q}} = 0 \)
  - \( V \) is radially unbounded.
  - \( \therefore \) Global asymptotic stability is ensured.
Example: Robot Manipulator

- In case, the gravitational terms is not canceled, $\dot{V}$ is modified to:

$$\dot{V} = -\dot{q}^T((K_D + B)\dot{q} + g(q))$$

- The presence of gravitational term means PD control alone cannot guarantee asymptotic tracking.

- Assuming that the closed loop system is stable, the robot configuration $q$ will satisfy

$$K_P(q_d - q) = g(q)$$

- The physical interpretation of the above equation is that:
  - The configuration $q$ must be such that the motor generates a steady state "holding torque" $K_P(q_d - q)$ sufficient to balance the gravitational torque $g(q)$.

∴ the steady state error can be reduced by increasing $K_P$. 
Control Design Based on Lyapunov’s Direct Method

- Basically there are two approaches to design control using Lyapunov’s direct method
  - Choose a control law, then find a Lyap. fcn to justify the choice
  - Candidate a Lyap. fcn, then find a control law to satisfy the Lyap. stability conditions.
- Both methods have a trial and error flavor
- In robot manipulator example the first approach was applied:
  - First a PD controller was chosen based on physical intuition
  - Then a Lyap. fcn. is found to show g.a.s.
Control Design Based on Lyapunov’s Direct Method

- Example: Regulator Design

- Consider the problem of stabilizing the system:
  \[ \ddot{x} - \dot{x}^3 + x^2 = u \]

- In other word, make the origin an asymptotically stable Equ. pt.

- Recall the example:
  \[
  \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -g(x_1) - h(x_2)
  \end{align*}
  \]

where \( g(.) \) & \( h(.) \) are locally Lip. and satisfy
\[
\begin{align*}
g(0) &= 0, \quad yg(y) > 0 \quad \forall y \neq 0, \quad y \in (-a, a) \\
h(0) &= 0, \quad yh(y) > 0 \quad \forall y \neq 0, \quad y \in (-a, a)
\end{align*}
\]

- Asymptotic stability of such system could be shown by selecting the following Lyap. fcn:
  \[
  V(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} g(y)dy
  \]
Example: Regulator Design

Let $x_1 = x$, $x_2 = \dot{x}$. The above example motivates us to select the control law $u$ as

$$u = u_1(\dot{x}) + u_2(x)$$

where

$$\dot{x}(\dot{x}^3 + u_1(\dot{x})) < 0 \text{ for } \dot{x} \neq 0$$
$$x(u_2(x) - x^2) < 0 \text{ for } x \neq 0$$

The globally stabilizing controller can be designed even in the presence of some uncertainties on the dynamics:

$$\ddot{x} + \alpha_1 \dot{x}^3 + \alpha_2 x^2 = u$$

where $\alpha_1$ and $\alpha_2$ are unknown, but s.t. $\alpha_1 > -2$ and $|\alpha_2| < 5$

This system can be globally stabilized using the control law:

$$u = -2\dot{x}^3 - 5(x + x^3)$$
Estimating Region of Attraction

- Sometimes just knowing a system is a.s. is not enough. At least an estimation of RoA is required.
  - Example: Occurring fault and finding "critical clearance time"

- Let $x = 0$ be an Equ. pt. of $\dot{x} = f(x)$. Let $\phi(t, x)$ be the sol starting at $x$ at time $t=0$. The Region Of Attraction (RoA) of the origin denoted by $R_A$ is defined by:

$$R_A = \{ x \in \mathbb{R}^n | \phi(t, x) \rightarrow 0 \text{ as } t \rightarrow \infty \}$$

- **Lemma:** If $x = 0$ is an a.s. Equ. pt. of $\dot{x} = f(x)$, then its RoA $R_A$ is an open, connected, invariant set. Moreover, the boundary of RoA, $\partial R_A$, is formed by trajectories of $\dot{x} = f(x)$.

- \[ \therefore \] one way to determine RoA is to characterize those trajectories that lie on $\partial R_A$. 
Example: Van-der-Pol

Dynamics of oscillator in reverse time

\[
\begin{align*}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= x_1 + (x_1^2 - 1)x_2
\end{align*}
\]

The system has an Equ. pt at origin and an unstable limit cycle.

The origin is a stable focus \(\longrightarrow\) it is a.s.
Example: Van-der-Pol

- Checking by linearization method

\[ A = \left. \frac{\partial f}{\partial x} \right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \]

- \( \lambda = -1/2 \pm j\sqrt{3}/2 \implies Re \lambda_i < 0 \)
- Clearly, RoA is bounded since trajectories outside the limit cycle drift away from it
- \( \therefore \) \( \partial R_A \) is the limit cycle
Example 11:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \frac{1}{3}x_1^3 - x_2
\end{align*}
\]

- There are 3 isolated Equ. pts. \((0, 0), (\sqrt{3}, 0), (-\sqrt{3}, 0)\).

- \((0, 0)\) is a stable focus, the other two are saddle pts.

- \(\therefore\) Origin is a.s. and other two are unstable (follows from linearization).

- Stable trajectories of the saddle points form two separatrices that are \(\partial R_A\).

- RoA is unbounded.
Example 11:

Recall the example:
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -h(x_1) - ax_2
\end{align*}
\]

\[
V = \frac{\delta}{2}x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \delta \int_0^{x_1} hdy
\]

Let
\[
V = \frac{1}{2}x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \int_0^{x_1} (y - \frac{1}{3}y^3) dy
\]

\[
= \frac{3}{4}x_1^2 - \frac{1}{12}x_1^4 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2^2
\]

We get: \[
\dot{V} = -\frac{1}{2}x_1^2 (1 - \frac{1}{3}x_1^2) - \frac{1}{2}x_2^2
\]

Define \( D = \{ x \in \mathbb{R}^2 | -\sqrt{3} < x_1 < \sqrt{3} \} \)

\[
\therefore V(x) > 0 \text{ and } \dot{V}(x) < 0 \text{ in } D - \{0\},
\]

From the phase portrait \( \Rightarrow D \) is not a subset of \( \mathbb{R}_A \). Tell me why?!!
Estimating RoA

- Traj starting in $D$ move from one Lyap. surface to $V(x) = c_1$ to an inner surface $V(x) = c_2$ with $c_2 < c_1$.

- However, there is no guarantee that the traj. will remain in $D$ forever.

- Once, the traj leaves $D$, no guarantee that $\dot{V}$ remains negative.

- This problem does not occur in $R_A$ since $R_A$ is an invariant set.

- The simplest estimate is given by the set
  \[ \Omega_c = \{ x \in \mathbb{R}^n | V(x) \leq c \} \]

  where $\Omega_c$ is bounded and connected and $\Omega_c \in D$

- Note that $\{ V(x) \leq c \}$ may have more than one component, only the bounded component which belong to $D$ is acceptable.

  - Example: If $V(x) = x^2/(1 + x^4)$ and $D = \{|x| < 1\}$

  - The set $\{ V(x) \leq 1/4 \}$ has two components $\{|x| \leq \sqrt{2 - \sqrt{3}}\}$ and $\{|x| \leq \sqrt{2 + \sqrt{3}}\} \rightsquigarrow$ only $\{|x| \leq \sqrt{2 - \sqrt{3}}\}$ is acceptable.
Estimating RoA

- To find RoA, first we need to find a domain $D$ in which $\dot{V}$ is n.d.
- Then, a bounded set $\Omega_c \subset D$ shall be sought
- We are interested in largest set $\Omega_c$, i.e. the largest value of $c$ since $\Omega_c$ is an estimate of $R_A$.
- $V$ is p.d. everywhere in $R^2$.
- If $V(x) = x^TPx$, let $D = \{x \in R^2 | \|x\| \leq r\}$. Once, $D$ is obtained, then select $\Omega_c \subset D$ by $c < \min_{\|x\|=r} V(x)$
- In words, the smallest $V(x) = c$ which fits into $D$.
- Since

$$x^TPx \geq \lambda_{min}(P)\|x\|^2$$

- We can choose

$$c < \lambda_{min}(P)r^2$$

- To enlarge the estimate of $R_A \Rightarrow$ find largest ball on which $\dot{V}$ is n.d.
Example 12:

\[
\begin{align*}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= x_1 + (x_1^2 - 1)x_2
\end{align*}
\]

- From the linearization \( \frac{\partial f}{\partial x} \bigg|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \) is stable
- Taking \( Q = I \) and solve the Lyap. equation:
  \[
  PA + A^T P = -I \implies P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}
  \]
- \( \lambda_{\min}(P) = 0.69 \)
- \( \dot{V} = -(x_1^2 + x_2^2) - (x_1^3x_2 - 2x_1^2x_2^2) \leq -\|x\|^2 + |x_1||x_1x_2||x_1 - 2x_2| \leq -\|x\|^2 + \frac{\sqrt{5}}{2}\|x\|^4 
  \]
  - where \( |x_1| \leq \|x\|_2, |x_1x_2| \leq \|x\|^2/2, |x_1 - 2x_2| \leq \sqrt{5}\|x\|_2 \)
- \( \dot{V} \) is n.d. on a ball \( D \) of radius
  \[
  r^2 = 2/\sqrt{5} = 0.894 \implies c < 0.894 \times 0.69 = 0.617
  \]
Example 12:

- To find less conservative estimate of $\Omega_c$:
- Let $x_1 = \rho \cos \theta$, $x_2 = \rho \sin \theta$

\[
\dot{V} = -\rho^2 + \rho^4 \cos^2 \theta \sin \theta (2\sin \theta - \cos \theta) \\
\leq -\rho^2 + \rho^4 |\cos^2 \theta \sin \theta| |2\sin \theta - \cos \theta| \\
\leq -\rho^2 + \rho^4 (.3849)(2.2361) \\
\leq -\rho^2 + .861 \rho^4 < 0 \text{ for } \rho^2 < \frac{1}{.861}
\]

- $c = .8 < \frac{.69}{.861} = .801$
- Thus the set:

\[
\Omega_c = \{x \in R^2 | V(x) \leq .8 \} \text{ is an estimate of } R_A.
\]
Example 12:

- A lesser conservative estimation of RoA:
  - plot the contour of $\dot{V} = 0$
  - plot $V(x) = c$ for increasing $c$ to find largest $c$ where $\dot{V} < 0$
- The $c$ obtained by this method is $c = 2.25$. 

![Graphs showing contours and comparison of region of attraction]

(a) Contours of $\dot{V}(x) = 0$ (dashed), $V(x) = 0.8$ (dash-dot), and $V(x) = 2.25$ (solid)

(b) Comparison of the region of attraction with its estimate.
Example 13:

\[
\begin{align*}
\dot{x}_1 &= -2x_1 + x_1 x_2 \\
\dot{x}_2 &= -x_2 + x_1 x_2
\end{align*}
\]

- There are two Equ. pts., \( (0, 0) \) and \( (1, 2) \).
- At \( (1, 2) \): \( A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \Rightarrow \text{unstable} \) \( (\lambda_{1,2} = \pm \sqrt{2}) \) (saddle pt.)
- At \( (0, 0) \): \( A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{a.s.} \)
- Taking \( Q = I \) and solving Lyap Eq. \( A^T P + PA = -I \Rightarrow P = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix} \)
- \( \therefore \) The Lyap. fcn is \( V(x) = x^T P x \)
- We have \( \dot{V} = -(x_1^2 + x_2^2) + \frac{1}{2}(x_1^2 x_2 + 2x_1 x_2^2) \)
- Find largest \( D \) s.t. \( \dot{V} \) is n.d. in \( D \).
- Let \( x_1 = \rho \cos \theta, \ x_2 = \rho \sin \theta \)
Example 13:

\[ \dot{V} = -\rho^2 + \rho^3 \cos\theta \sin\theta \left( \sin\theta + \frac{1}{2} \cos\theta \right) \]

\[ \leq -\rho^2 + \frac{1}{2} \rho^3 |\sin 2\theta| \left| \sin\theta + \frac{1}{2} \cos\theta \right| \]

\[ \leq -\rho^2 + \frac{\sqrt{5}}{4} \rho^3 < 0 \quad \text{for} \quad \rho < \frac{4}{\sqrt{5}} \]

- Since \( \lambda_{\min}(P) = \frac{1}{4} \implies \), we choose
  \[ c = .79 < \frac{1}{4} \times \left( \frac{4}{\sqrt{5}} \right)^2 = .8 \]

- Thus the set:
  \[ \Omega_c = \{ x \in R^2 | V(x) \leq .79 \} \subset R_A. \]

- Estimating RoA by the set \( \Omega_c \) is simple but conservative

- Alternatively Lasalle’s theorem can be used. It provides an estimate of \( R_A \).
Example 14:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -4(x_1 + x_2) - h(x_1 + x_2)
\end{align*}
\]

where \( h : \mathbb{R} \rightarrow \mathbb{R} \) s.t. \( h(0) = 0 \), \( \& xh(x) \geq 0 \quad \forall |x| \leq 1 \)

\( \quad \nabla \) Consider the Lyap fcn candidate:

\[
V(x) = x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x = 2x_1^2 + 2x_1x_2 + x_2^2
\]

\( \quad \nabla \) Then \( \dot{V} = -2x_1^2 - 6(x_1 + x_2)^2 - 2(x_1 + x_2)h(x_1 + x_2) \leq \)

\[
-2x_1^2 - 6(x_1 + x_2)^2 = -x^T \begin{bmatrix} 8 & 6 \\ 6 & 6 \end{bmatrix} x, \quad \forall |x_1 + x_2| \leq 1
\]

\( \quad \nabla \) \( \dot{V} \) is n.d. in the set \( G = \{ x \in \mathbb{R}^2 | |x_1 + x_2| \leq 1 \} \).

\( \quad \nabla \) \( (0, 0) \) is a.s. to estimate \( R_A \), first do it from \( \Omega_c \).
Example 14:

Find the largest $c$ s.t. $\Omega_c \subset G$. Now, $c$ is given by

$$c = \min_{|x_1+x_2|=1} V(x) \text{ or }$$

$$c = \min \left\{ \min_{x_1+x_2=1} V(x), \min_{x_1+x_2=-1} V(x) \right\}$$

The first minimization yields

$$\min_{x_1+x_2=1} V(x) = \min_{x_1} \left\{ 2x_1^2 + 2x_1(1-x_1) + (1-x_1)^2 \right\} = 1$$

and

$$\min_{x_1+x_2=-1} V(x) = 1$$

Hence, $\Omega_c$ with $c = 1$ is an estimate of $R_A$.

A better (less conservative) estimate of $R_A$ is possible.
Example 14:

- The key point is to observe that traj inside $G$ cannot leave it through certain segment of the boundary $|x_1 + x_2| = 1$.
- Let $\sigma = x_1 + x_2 \implies \partial G$ is given by $\sigma = 1$ and $\sigma = -1$
- We have
  \[
  \frac{d}{dt} \sigma^2 = 2\sigma(\dot{x}_1 + \dot{x}_2) = 2\sigma x_2 - 8\sigma^2 - 2\sigma h(\sigma) \\
  \leq 2\sigma x_2 - 8\sigma^2, \quad \forall |\sigma| \leq 1
  \]
- On the boundary $\sigma = 1 \implies \frac{d\sigma^2}{dt} \leq 2x_2 - 8 \leq 0 \quad \forall x_2 \leq 4$
- Hence, the traj on $\sigma = 1$ for which $x_2 \leq 4$ cannot move outside the set $G$ since $\sigma^2$ is non-increasing
- Similarly, on the boundary $\sigma = -1$ we have
  \[
  \frac{d\sigma^2}{dt} \leq -2x_2 - 8 \leq 0 \quad \forall x_2 \geq -4
  \]
- Hence, the traj on $\sigma = -1$ for which $x_2 \geq -4$ cannot move outside the set $G$. 
Example 14:

- To define the boundary of $G$, we need to find two other segments to close the set.
- We can take them as the segments of Lyap. fcn surface
- Let $c_1$ be s.t. $V(x) = c_1$ intersects the boundary of $x_1 + x_2 = 1$ at $x_2 = 4$ and let $c_2$ be s.t. $V(x) = c_2$ intersects the boundary of $x_1 + x_2 = -1$ at $x_2 = -4$
- Then, we define $V(x) = \min c_1, c_2$, we have

$$c_1 = V(x)|_{x_1 = -3, x_2 = 4} = 10 \quad \& c_2 = V(x)|_{x_1 = 3, x_2 = -4} = 10$$
Example 14:

- The set $\Omega$ is defined by

$$\Omega = \{ x \in R^2 | V(x) \leq 10 \ \& \ |x_1 + x_2| \leq 1 \}$$

- This set is closed and bounded and **positively invariant**. Also, $\dot{V}$ is n.d. in $\Omega$ since $\Omega \subset G \implies \Omega \subset R_A$. 

![Graph showing estimates of the region of attraction](image-url)