# Computational Intelligence
# Lecture 4:Linguistic Variables and Fuzzy Rules

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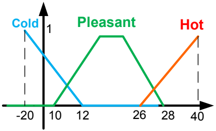
# Linguistic Variables

- Linguistic Variable: when a variable can take words in natural languages as its values.
  - the words are characterized by fuzzy sets defined in the universe of discourse
- **Example:** The weather temperature: a variable $x \in [-20\ 40]$
  - Let's define three fuzzy sets "Cold" "Pleasant," and "Hot"
  - $x$: a linguistic variable, $\rightsquigarrow$ "x is cold,"
  - $x$ also can take numbers in $[-20\ 40] \rightsquigarrow, x = 10^o C$



- Linguistic Variable: is characterized by $(X, T, U, M)$
  - $X$: the name of the linguistic variable (the weather temperature)
  - $T$: the set of linguistic values that $X$ can take ($T = \{cold, pleasant, hot\}$)
  - $U$: the actual physical domain in which the linguistic variable $X$ takes its quantitative (crisp) values ($U = [-20, 40]$).
  - $M$: a semantic rule that relates each linguistic value in $T$ with a fuzzy set in $U$; ($M$ relates "cold," "pleasant," and "hot" with the membership functions shown in Fig)
- Linguistic Hedge:
  - One may use more than one word to describe the ling. var. ("very cold", "not pleasant", "more or less hot")
  - The value of a ling. var. is a composite of atomic term $x = x_1 x_2 \ldots x_n$
  - The atomic terms can be classified into:
    - Primary terms: labels of fuzzy sets; "cold," "pleasant," and "hot."
    - Complement: "not", "and" and "or."
    - Hedges: "very," "slightly," "more or less,"

# Some Examples of Linguistic Hedge

- Let $A$ be a fuzzy set in $U$
  - very $A$ : a fuzzy set in $U$ $\mu_{verry\ A}(x) = [\mu_A(x)]^2$
  - more or less $A$: a fuzzy set in $U$ $\mu_{more\ or\ less\ A}(x) = [\mu_A(x)]^{1/2}$
- **Example:** $U = \{10, \ldots, 130\}$
  - fuzzy set "fast" $= 1/130 + 1/120 + 0.8/100 + 0.6/80 + 0.1/40$
  - "very very fast" $= 1/130 + 1/120 + 0.4096/100 + 0.1296/80 + 0.0001/40$
  - "more or less fast" $= 1/130 + 1/120 + 0.8944/100 + 0.7746/80 + 0.3162/40$

# Fuzzy IF-Then Rules

- **IF <fuzzy proposition>, THEN < fuzzy proposition>**
- There are two types of fuzzy proposition
  - Atomic fuzzy proposition: a single statement
    $x$ is $A$
  - Compound fuzzy proposition: a composition of atomic fuzzy propositions using the connectives "and," "or," and "not"
    $x$ is not $S$ and $x$ is not $F$
  - $S, F, A$ are fuzzy sets
- Compound fuzzy propositions should be considered as fuzzy relations.

- Let $x, y$: linguistic variables in the physical domains $U, V$, and $A, B$: fuzzy sets in $U, V$
  - Connective "and" *x is A and y is B* use fuzzy intersection:
    - $A \bigcap B \in U \times V$: $\mu_{A \bigcap B}(x, y) = t[\mu_A(x), \mu_B(y)]$
    - $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is any t-norm.
  - Connective "or" *x is A or y is B* use fuzzy union:
    - $A \bigcup B \in U \times V$: $\mu_{A \bigcup B}(x, y) = s[\mu_A(x), \mu_B(y)]$
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  - Connective "not" *x is not A* use fuzzy complements.

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  - Connective "not" *x is not A* use fuzzy complements.
- **Example:** Recall the weather example:
  - $HO = (x \text{ is } C \text{ and } x \text{ is not } H) \text{ or } x \text{ is } P$: a fuzzy relation in the product space $[-20, 40]$:
  - $C = cold, H = hot, P = pleasant$, $x_1 = x_2 = x_3 = x$
  - $\mu_{HO}(x_1, x_2, x_3) = s\{t[\mu_c(x_1), c(\mu_H(x_2))], \mu_M(x_3)\}$

# Interpretation of Fuzzy Rules

- For classical rules:
  - **IF** $p$ **THEN** $q$ $\rightsquigarrow p \rightarrow q$
  - $p \rightarrow q \equiv \bar{p} \bigvee q \equiv \bar{p} \bigvee (p \bigwedge q)$
- For fuzzy rules
  - $p$ and $q$ are fuzzy propositions
  - $\bar{.} \equiv$ complement; $\bigvee \equiv$ s-norm; $\bigwedge \equiv$ t-norm
  - Different c-norm, s-norm, t-norm yields verity of interpretations
- *IF* $<$ *FP1* $>$ *THEN* $<$ *FP2* $>$
  - $FP1, FP2$: fuzzy relations in $U = U_1 \times ... \times U_n, V = V_1 \times \dots V_m$,
  - $x$ and $y$ are linguistic variables (vectors) in $U$ and $V$
- Some popular Fuzzy interpretation:
  - **Dienes-Rescher Implication:** Using basic fuzzy c-norm and s-norm, $Q_D \in U \times V$
    $\mu_{Q_d}(x, y) = \max[1 - \mu_{FP1}(x), \mu_{FP2}(y)]$

# Interpretation of Fuzzy Rules

- **Lukasiewicz Implication:** Using basic fuzzy c-norm and Yager s-norm with $w = 1$, $Q_L \in U \times V$
  $\mu_{Q_L}(x, y) = \min[1, 1 - \mu_{FP1}(x) + \mu_{FP2}(y)]$
- **Zadeh Implication:** Using basic fuzzy c-norm, s-norm, and t-norm, $Q_Z \in U \times V$
  $\mu_{Q_Z}(x, y) = \max\{\min[\mu_{FP1}(x), \mu_{FP2}(y)], 1 - \mu_{FP1}(x)\}$
- **Gödel Implication:** a well-known implication in classical logic, $Q_G \in U \times V$
  $$\mu_{Q_G}(x, y) = \begin{cases} 1 & \textit{if } \mu_{FP1}(x) \leq \mu_{FP2}(y) \\ \mu_{FP2}(y) & \textit{otherwise} \end{cases}$$

- **Lemma:** For all $(x, y) \in U \times V$
  $\mu_{Q_z}(x, y) \leq \mu_{Q_D}(x, y) \leq \mu_{Q_L}(x, y)$
- **Proof:**
  - $0 \leq 1 - \mu_{FP1}(x) \leq 1$ and
    $0 \leq \mu_{FP2}(y) \leq 1 \rightsquigarrow \max\{1 - \mu_{FP1}(x), \mu_{FP2}(y)\} \leq 1 - \mu_{FP1}(x) + \mu_{FP2}(x)$
  - $\max\{1 - \mu_{FP1}(x), \mu_{FP2}(y)\} \leq 1 \rightsquigarrow \mu_{Q_D} \leq \mu_{Q_L}$
  - $\min[\mu_{FP1}(x), \mu_{FP2}(y)] \leq \mu_{FP2}(y) \rightsquigarrow \mu_{Q_z}(x, y) \leq \mu_{Q_D}(x, y)$
- We will learn later, the criteria to choose the combination of c-norms, t-norms, and s-norms for interpretation
- In crisp logic $p \rightarrow q$ is global implication,
  - It covers all possible choices
- In fuzzy logic $p \rightarrow q$ is local implication
  - Example: *IF temperature is high THEN turn on the air condition*
  - no guideline when "temperature is medium" , or "temperature is low"
- $\therefore$ the fuzzy rule should be mentioned as:
  $IF < FP_1 > THEN < FP_2 > ELSE < NOTHING > \rightsquigarrow p \rightarrow q = p \bigwedge q$

- using min or algebraic product leads to **Mamdani implications** (the most popular implication)
  $\mu_{Q_{MM}}(x, y) = \min[\mu_{FP1}(x), \mu_{FP2}(y)]$
  $\mu_{Q_{MP}}(x, y) = \mu_{FP1}(x)\mu_{FP2}(y)$
- BUT some rule may implicitly clarify the else part
  - Example: *IF temperature is high THEN turn on the air condition*
  - It implicitly mentions that *IF temperature is low THEN turn off the air condition*
- diff. people have diff. interpretations.
- $\therefore$ diff. implications are required
- If the human experts think that their rules are local, then use the Mamdani implications; otherwise, use the global implications

## Example

- Let $U = \{-10, 5, 15, 25\}$, $V = \{1, 2, 3\}$
- $x \in U$: temperature, $y \in V$: air condition mode
- IF-THEN rule: *IF x is low THEN y is small*
- fuzzy set "high" $= 1/-10 + .7/5 + .1/15 + 0/25$
- fuzzy set "small" $= 1/1 + 0.5/2 + 0.1/3$
- $Q_D = 1/(-10,1) + 0.5/(-10,2) + 0.1/(-10,3) + 1/(5,1) + 0.5/(5,2) + 0.3/(5,3) + 1/(15,1) + 0.9/(15,2) + 0.9/(15,3) + 1/(25,1) + 1/(25,2) + 1/(25,3)$
- $Q_L = 1/(-10,1) + 0.5/(-10,2) + 0.1/(-10,3) + 1/(5,1) + 0.8/(5,2) + 0.4/(5,3) + 1/(15,1) + 1/(15,2) + 1/(15,3) + 1/(25,1) + 1/(25,2) + 1/(25,3)$
- $Q_Z = 1/(-10,1) + 0.5/(-10,2) + 0.1/(-10,3) + 0.7/(5,1) + 0.5/(5,2) + 0.3/(5,3) + 0.9/(15,1) + 0.9/(15,2) + 0.9/(15,3) + 1/(25,1) + 1/(25,2) + 1/(25,3)$

# Example: Cont'd

- $Q_G = 1/(-10,1) + 0.5/(-10,2) + 0.1/(-10,3) + 1/(5,1) + 0.5/(5,2) + 0.1/(5,3) + 1/(15,1) + 1/(15,2) + 1/(15,3) + 1/(25,1) + 1/(25,2) + 1/(25,3)$
- $Q_{MM} = 1/(-10,1) + 0.5/(-10,2) + 0.1/(-10,3) + 0.7/(5,1) + 0.5/(5,2) + 0.1/(5,3) + 0.1/(15,1) + 0.1/(15,2) + 0.1/(15,3) + 0/(25,1) + 0/(25,2) + 0/(25,3)$
- $Q_{MM} = 1/(-10,1) + 0.5/(-10,2) + 0.1/(-10,3) + 0.7/(5,1) + 0.35/(5,2) + 0.07/(5,3) + 0.1/(15,1) + 0.05/(15,2) + 0.01/(15,3) + 0/(25,1) + 0/(25,2) + 0/(25,3)$
- membership of $(25,1),(25,2),(25,3)$ is full for $Q_z, Q_D, Q_L, Q_G$,
- It is zero for $Q_{MM}$ and $Q_{MP}$ since $\mu_{high}(25) = 0$

- ∴ Dienes-Rescher, Lukasiewicz, Zadeh and Godel implications are global,
- Mamdani implications are local.
- BUT Manipulations are easier with Mamdani implications rather than others