Nonlinear Control
Lecture 3: Fundamental Properties

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- The norm $\|x\|$ of a vector x is a real-valued function s.t.
  1. $\|x\| \geq 0 \; \forall x \in R^n, \quad \|x\| = 0$ iff $x = 0$
  2. $\|x + y\| \leq \|x\| + \|y\|, \; \forall x, y \in R^n$
  3. $\|ax\| = |a|\|x\| \;\; \forall a \in R, \; \forall x \in R^n$
- The class $p - norm, \; p \in [1, \infty)$ are defined by

$$\|x\|_p = (|x_1|^p + ... + |x_n|^p)^{1/p}$$

- $\|x\|_\infty = \max_i |x_i|$
- The three most common norms are:
  $\|x\|_1, \; \|x\|_\infty$, and the Euclidean norm $\|x\|_2 = \left(x^T x\right)^{1/2}$
- All p-norms are equivalent in the sense that $\exists c_1$ & c_2 s.t.:

$$c_1\|x\|_\alpha \leq \|x\|_\beta \leq c_2\|x\|_\alpha \;\; \forall x \in R^n$$

e.g.: $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$
$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$
$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$

- An $m \times n$ matrix A defines a linear mapping $y = Ax$ from R^n into R^m. The induced $p-norm$ of A is defined by:

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\|x\| \leq 1} \|Ax\|_p = \sup_{\|x\|=1} \|Ax\|_p$$

- for $p = 1, 2, \infty$, we have
  $\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|$
  $\|A\|_2 = \sigma_{max}(A) = \left[\lambda_{max}(A^T A)\right]^{1/2}$
  $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$
- we have
  $\|A\|_2 \leq \sqrt{\|A\|_1 \|A\|_\infty}$
  $\frac{1}{n}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty$
  $\frac{1}{m}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\|A\|_1$
  $\|AB\|_p \leq \|A\|_p \|B\|_p$
- **Hölder inequality:**
  $|x^T y| \leq \|x\|_p \|y\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad x, y \in R^n$

Set

- A set S is closed iff every convergent sequence x_d with elements in S converges to a point in S.
- A set S is bounded if there is $r > 0$ s.t. $\|x\| \leq r$ for all $x \in S$.
- A set S is compact if it is closed and bounded.
- A set S is convex: if for every $x, y \in S$ and every real number $\theta,\ 0 < \theta < 1$, the point $\theta x + (1 - \theta)y \in S$.

Continuous Function

- A function f mapping a set S_1 into a set S_2 is denoted by $f : S_1 \rightarrow S_2$.
- f is continuous at x if, given $\epsilon > 0$, there is $\delta > 0$ s.t

$$\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \epsilon \tag{1}$$

- A function f is continuous on a set S if it is continuous at every point of S
- A function f is uniformly continuous on S if given $\epsilon > 0$ there is $\delta > 0$ (dependent only on ϵ) s.t. (1) holds for all $x, y \in S$.
- For uniform continuity, the same constant δ works for all points in the set.
- f is uniformly continuous on a set $S \Rightarrow$ it is continuous on S. But the opposite is not true in general.
- If S is a compact set, then continuity $\equiv$ uniform continuity.

Continuous Differentiable Function

- A function $f : R \rightarrow R$ is differentiable at x if

$$\acute{f}(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- A function $f : R^n \rightarrow R^m$ is continuously differentiable at a point x_0 if $\frac{\partial f_i}{\partial x_j}$ exist and are continuous at x_0 for $1 \leq i \leq m, \ 1 \leq j \leq n$.
- A function f is continuously differentiable on a set S if it is continuously differentiable at every point of S.

Mean Value

- If x and y are two distinct points in R^n, then the line segment $L(x, y)$ joining x and y is given by:

$$L(x, y) = \{z = \theta x + (1 - \theta)y, \quad 0 < \theta < 1\}$$

- Mean Value Theorem: *Assume $f : R^n \longrightarrow R$ is continuously differentiable at each point x on an open set $S \subset R^n$. Let x and y be two points of S s.t. the line segment $L(x, y) \subset S$. Then, there exists a point z of $L(x, y)$ s.t.:*

$$f(y) - f(x) = \left.\frac{\partial f}{\partial x}\right|_{x=z} (y - x)$$

  - Proof in: T. M. Apostol. Mathematical Analysis. Addison-Wesley, Reading, MA, 1957.

Existence

- This section provides sufficient condition for uniqueness and existence solution of the initial value problem

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{2}$$

- Existence of solution is provided by continuity:
- A solution of (2) over an interval $[t_0, t_1]$:
  $x : [t_0, t_1] \longrightarrow R^n$ s.t. $\dot{x}(t)$ is defined, $\dot{x}(t) = f(t, x(t))\ \forall t \in [t_0, t_1]$
  - If f is continuous in t and $x \rightsquigarrow$ the solution $x(t)$ is continuously differentiable.
  - Assume f is continuous in x but only *piecewise continuous* in $t \rightsquigarrow x(t)$ is *only piecewise continuously differentiable*.
- This allows time-varying input with step changes in time.

Existence

- A differential equation might have many solutions, e.g.

$$\dot{x} = x^{1/3}, \qquad x(0) = 0 \tag{3}$$

- $x(t) = (2t/3)^{3/2}$ and $x(t) = 0 \rightsquigarrow$ the solution is not unique.
- However, f is continuous $\rightsquigarrow$ continuity is not sufficient to ensure uniqueness.
- Continuity of f guarantees *at least* one solution.

Existence and Uniqueness

- **Theorem 3.1 (Lipschitz condition: Local Existence and Uniqueness)** *Let $f(t,x)$ be piecewise continuous in t and satisfy the Lipschitz condition:*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\| \quad \forall x, y \in B = \{x \in R^n \| \|x - x_0\| \leq r\}, \quad \forall t \in [t_0, t_1]$$

Then, there exists $\delta > 0$ such that the state equation $\dot{x} = f(t,x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

  - The function f satisfying Lipschitz condition is called Lipschitz in x
  - The constant L is called the Lipschitz constant.
  - A function can be locally or globally Lipschitz.

Existence and Uniqueness

- A function $f(x)$ is said to be **locally Lipchitz** on a domain $D \subset R^n$ (open and connected set) if each point of D has a neighborhood D_0 such that $f(x)$ satisfies the Lipschitz condition for all points on D_0 with **some** Lipschitz constant L_0.
- A function $f(x)$ is set to be Lipchitz on a set W if it satisfies Lipschitz condition for all points with the **same** Lipschitz constant.
- $\therefore$ A locally Lipschitz function on D is not necessarily Lipschitz on D since the Lipschitz condition may not hold uniformly (with the same Lipschitz constant) for all points in D.
- A function $f(x)$ is said to be **globally Lipchitz** if it is Lipschitz on R^n.

Existence and Uniqueness

- The same terminology holds for $f(t,x)$ if the Lipchitz condition is hold uniformly in t for all t in a certain interval.
- A function $f(t,x)$ is said to be locally Lipchitz on $[a,b] \times D \subset R \times R^n$ if each point of $x \in D$ has a neighborhood D_0 such that $f(t,x)$ satisfies the Lipschitz condition for **same** Lipschitz constant L_0 on $[a,b] \times D_0$.
- If f is scalar, $f : R \longrightarrow R$, the Lipschitz condition can be expressed as:

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L$$

- The line connecting every two points of f, cannot have a slope $>$ L.
- $\therefore$ If a function has infinite slope at some points, the function cannot be locally Lipschitz at those points.
- Discontinuous functions cannot be locally Lipschitz at the points of discontinuity.

Existence and Uniqueness

- **Example:** $f(x) = x^{1/3}$ is not locally Lip. at $x = 0$ since $f'(x) = (1/3)x^{-2/3} \longrightarrow \infty$ as $x \longrightarrow 0$.
  - If $f'(x)$ in some region is bounded by k, then f is lip on that region with Lip. constant $L = k$.
- This fact is also true for vector valued functions
- **Lemma 3.1**: *Let $f : [a, b] \times D \longrightarrow R^m$ be continuous for some domain $D \in R^n$. If for a convex subset $W \subset D$ there is a constant $L \geq 0$ s.t.*

$$\|\frac{\partial f}{\partial x}(t, x)\| \leq L \qquad \text{on } [a, b] \times W,$$

then $\|f(t, x) - f(t, y)\| \leq L\|x - y\|$ for all $t \in [a, b]$, $x \in W$, and $y \in W$.

Existence and Uniqueness

- **Proof:**
- Let $\|.\|_p$ be any norm $p \in [1, \infty]$ and determine q s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Fix t on $[a, b]$ and assume $x \in W$, $y \in W$.
- Define $\gamma(s) = (1-s)x + sy, \quad s \in R, \quad \gamma(s) \in D,$
- $W \subset D$ is convex $\rightsquigarrow$ $\gamma(s) \in W$ for $0 \le s \le 1$.
- Take $z \in R^m$ s.t.

$$\|z\|_q = 1, \quad z^T[f(t,y) - f(t,x)] = \|f(t,y) - f(t,x)\|_p$$

- set $g(s) = z^T f(t, \gamma(s))$. Since, $g(s)$ is a continuously differentiable real-valued function over the open interval which includes $[0, 1]$, from mean-value theorem , $\exists s_1 \in (0, 1)$ s.t.

$$g(1) - g(0) = g'(s_1)$$

Existence and Uniqueness

- Evaluating g at $s = 0$ and $s = 1$:

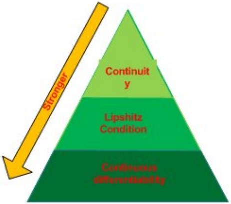
$$z^T[f(t,y) - f(t,x)] = z^T \frac{\partial f}{\partial x}(t,\gamma(s_1))(y-x)$$

- and using chain rule in calculating $g'(s)$ and Hölder inequality, $|z^T w| \leq \|z\|_q \|w\|_p$:

$$\|f(t,y) - f(t,x)\|_p \leq \|z\|_q \left\| \frac{\partial f}{\partial x}(t,\gamma(s_1)) \right\|_p \|y-x\|_p \leq L \|y-x\|_p$$

Existence and Uniqueness

- If f is Lip. on W, $\Rightarrow$ it is uniformly continuous on W, (prove it) but the converse is not true
- The function $f(x) = x^{1/3}$ is continuous on R, but it's not locally lip on $x = 0$.
- Lip. condition is weaker than continuous differentiability condition :



Existence and Uniqueness

- **Lemma 3.2** *If $f(t,x)$ and $[\frac{\partial f}{\partial x}](t,x)$ are continuous on $[a,b] \times D$ for some domain $D \subset R^n$, then f is locally Lip. in x on $[a,b] \times D$.*
- **Proof:**
  - For $x_0 \in D$, let r be so small that the ball $D_0 = \{x \in R^n | \|x - x_0\| \leq r\}$ is contained in D
  - The set D_0 is convex and compact
  - By continuity, $\frac{\partial f}{\partial x}$ is bounded on $[a,b] \times D_0$.
  - Let L_0 is a bound for $\frac{\partial f}{\partial x}$ on $[a,b] \times D_0$
  - By Lemma 3.1, $f(t,x)$ is Lip. on $[a,b] \times D_0$ with Lip. constant L_0.
- **Lemma 3.3**: *If $f(t,x)$ and $[\frac{\partial f}{\partial x}](t,x)$ are continuous on $[a,b] \times R^n$, then f is globally Lip. in x on $[a,b] \times R^n$ iff $[\frac{\partial f}{\partial x}]$ is uniformly bounded on $[a,b] \times R^n$.*
  - $x(t)$ is uniformly bounded if $\exists c > 0$, independent of $t_0 > 0$, and for every $a \in (0,c)$, there is $\beta = \beta(a) > 0$, independent of t_0, s.t.

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta, \forall t \geq t_0$$

Example 1

$$f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$$

- f is continuously differentiable on $R^2 \implies f$ is locally Lip. on R^2.
- f is not globally Lip. since $\frac{\partial f}{\partial x}$ is not uniformly bounded on R^2.
- However, it is Lip. on any compact set on R^2.
- Find the Lip. constant on set $W = \{x \in R^2 || x_1| \leq a_1, \ |x_2| \leq a_2\}$.
  - fist find jacobian matrix $\frac{\partial f}{\partial x} = \begin{bmatrix} -1 + x_2 & x_1 \\ -x_2 & 1 - x_1 \end{bmatrix}$
  - Use ∞ norm for vectors and induced norm for matrices:

$$\begin{aligned} \|\frac{\partial f}{\partial x}\|_\infty &= max\{|-1+x_2| + |x_1|, |x_2| + |1-x_1|\} \\ |-1+x_2| + |x_1| &\leq 1 + a_2 + a_1, \quad |x_2| + |1-x_1| \leq a_2 + 1 + a_1 \\ \|\frac{\partial f}{\partial x}\|_\infty &\leq 1 + a_1 + a_2 \rightsquigarrow L_0 = 1 + a_1 + a_2 \end{aligned}$$

Example 2

$$f(x) = \begin{bmatrix} x_2 \\ -sat(x_1 + x_2) \end{bmatrix}$$

- f is **not** continuously differentiable on R^2.
- Lip. condition is evaluated by definition.
- Use $\|.\|_2$ and also note that

$$\begin{aligned} |sat(\eta) - sat(\zeta)| &\leq |\eta - \zeta| \\ \therefore \|f(x) - f(y)\|_2 &\leq (x_2 - y_2)^2 + (x_1 + x_2 - y_1 - y_2)^2 \\ &= (x_1 - y_1)^2 + 2(x_1 - y_1)((x_2 - y_2) + 2(x_2 - y_2)^2 \end{aligned}$$

- We have

$$a^2 + 2ab + 2b^2 = \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \leq \lambda_{max} \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right\} \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|_2^2$$

- $\therefore \|f(x) - f(y)\|_2 \leq \sqrt{2.618}\|x - y\|_2, \quad \forall x, y \in R^2$

- If we use the more conservative inequality

$$a^2 + 2ab + 2b^2 \leq 2a^2 + 3b^2 \leq 3(a^2 + b^2)$$

- The Lip constant $\sqrt{3}$ is obtained.
- **Therefore**
  - Type of norm does not affect the Lip. property, but it does affect the Lip. constant
  - If the Lip. condition is satisfied for some L_0, it is also hold for all $L > L_0$.
  - Lip. constant is not unique
  - Theorem 3.1 is a local theorem
  - It guarantees the existence and uniqueness for the interval $[t_0, t_0 + \delta]$.
  - Existence and uniqueness for the interval $[t_0, t_1]$ is not clear.
- One way is to repeatedly apply the local theorem 3.1 and extend the existence interval
  - Start with t_0, x_0, the existence and uniqueness is guaranteed for $[t_0, t_0 + \delta]$.
  - Take new initial condition as $t_0 + \delta$ and $x(t_0 + \delta)$ and extend the interval to $[t_0 + \delta, t_0 + \delta + \delta_2]$.
  - Repeat the procedure

- In general, the procedure cannot go indefinitely
  - $\therefore$ there is a maximum interval $[t_0, T]$ that the unique solution that starts from (t_0, x_0) exists.
  - T might be smaller than t_1, in this case when $t \longrightarrow T$, the solution leaves the set on which f is locally Lip.
- **Example 3.3**

$$\dot{x} = -x^2, \quad x(0) = -1 \tag{4}$$

- f is locally Lip. for all $x \in R$.
- It is locally Lip. on all compact subset of R

$$x(t) = \frac{1}{t-1} \qquad \text{Unique solution} \quad \text{on } [0, 1]$$

- As $t \longrightarrow 1$ $x(t)$ leaves the set.
- Finite escape time indicates that the trajectories go to infinity in finite time.

When the solution interval can be extended indefinitely?

- On way is to guarantee that the solution $x(t)$ always remain in the set on which is uniformly Lip.
- This is achieved if function f is globally Lip.
- **Theorem 3.2 (Global Existence and Uniqueness)** *Suppose that $f(t,x)$ is piecewise continuous in t and satisfies*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\| \ \ \forall x, y \in R^n, \ \forall t \in [t_0, t_1]$$

Then, $\dot{x} = f(t,x), \ \ x(t_0) = x_0$ has a unique solution on $[t_0, t_1]$.

- **Example 3.4:** $\dot{x} = A(t)x + g(t) = f(t,x)$
- where $A(t)$ and $g(t)$ are piecewise continuous functions in t.
- Over any finite interval, elements of $A(t)$ and $g(t)$ are bounded

$$\|A(t)\| \leq a \ \text{ using any induced norm}$$

- Example 3.4. Contd.
  - All conditions of Theorem 3.2 is satisfied since $\forall x, y \in R^n$ and $t \in [t_0, t_1]$:

$$\|f(t,x) - f(t,y)\| = \|A(t)(x-y)\| \leq \|A(t)\|\|x-y\| \leq a\|x-y\|$$

  - Linear System has a unique solution over $[t_0, t_1]$.
  - t_1 can be arbitrarily large $\rightsquigarrow$ if $A(t)$ and $g(t)$ are piecewise continuous functions, system has a unique solution for $t \geq t_0$ and cannot have "finite escape time".
- The global Lip. condition is reasonable for linear systems.
- In general, it is rarely satisfied for nonlinear systems
- Local Lip. condition is essentially related to smoothness of f
- It is automatically satisfied if f it is continuously differentiable
- Except for hard nonlinearities which are idealization of nonlinear phenomena, physical system models satisfy Lip. condition
- Continuous functions which are not locally Lip. are rare in practice.
- However, the global Lip. condition cannot be satisfied by many physical

- Theorem 3.2 provides conservative condition on unique solution of nonlinear systems
- **Example 3.5:** $\dot{x} = -x^3 = f(x)$
  - $f(x)$ is not globally Lip. since Jacobian $\frac{\partial f}{\partial x}$ is not bounded in R.
  - However, for $x(t_0) = x_0$, the unique solution is given by

$$x(t) = sign(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2(t - t_0)}}$$

- By having some knowledge about the solution $x(t)$, one can proved less conservative condition for uniqueness using local Lip. condition on f

Summery

- Solution exitance for $\dot{x} = f(x,t)$ is achieved by continuity or at lease piecewise continuity of function f in t.
- Lipshitz condition can provide sufficient condition for unique solution
- Theorem 3.1: *Let $f(t,x)$ be piecewise continuous in t and satisfy the Lipschitz condition:*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\| \quad \forall x,y \in B = \{x \in R^n | \|x - x_0\| \leq r\}, \quad \forall t \in [t_0, t_1]$$

Then, there exists $\delta > 0$ such that the state equation $\dot{x} = f(t,x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

Summery

- Locally Lipshitz
  - The condition is satisfied on a subset $D \subset R^n$
  - It guarantees unique solution over $[t_0, t_0 + \delta]$
  - A function $f(x)$ is set to be Lipchitz on a set W if it satisfies Lipschitz condition for all points with the **same** Lipschitz constant.
  - To check the Lipshitz conation a convex subset $W \subset D$, it is sufficient to satisfy: $\|\frac{\partial f}{\partial x}(t, x)\| \leq L \quad \text{on } [a, b] \times W$.
  - To find Lip. constant, L, type of norm does not affect the Lip. property, but it does affect the Lip. constant.
  - Lip. constant is not unique.
  - Continuously differentiability of $f(t, x)$ on $[a, b] \times D$ guarantees f to be locally Lip.

Summery

- Globally Lipshitz
  - The condition is satisfied on R^n
  - It guarantees unique solution over $[t_0, t_1]$, (no matter how large t_1 is)
  - Continuously differentiability of $f(t,x)$+ uniformly boundedness of $\frac{\partial f}{\partial x}$ on $[a,b] \times R^n$ guarantees f to be globally Lip.
  - uniformly boundedness of $\frac{\partial f}{\partial x}$ is a killer condition and difficult to be satisfied for nonlinear systems in practice.
  - By having some knowledge about the solution x(t), we are looking for less conservative condition for uniqueness.

- **Theorem 3.3:** *Let $f(t,x)$ is piecewise continuous in t and is locally Lip. in x for all $t \geq t_0$ and all $x \in D \subset R^n$. Let W be a compact subset of D, $x_0 \in W$ and every solution of $\dot{x} = f(t,x), \quad x(t_0) = x_0$ lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$.*
- **Proof:**
  - The proof is based on the fact that if the solution remains in the set W, it cannot have "finite escape time".
  - By Theorem 3.1, the unique solution exist in the interval $[t_0, t_0 + \delta]$. From the previous discussion we know that if T is finite, the solution must leave D, however, since the solution never leaves W, we conclude that $T = \infty$.
- The problem in applying this theorem is to show that the solution never leaves the set W.
- We desire to check the assumption that every solution lies in a compact set without actually solving the differential equation.
- Lyapunov's stability theorem is an important tool for this purpose.

Example 3.6:

$$\dot{x} = -x^3 = f(x)$$

- $f(x)$ is locally Lip. on R
- $\begin{cases} x(t) > 0 & \Longrightarrow & \dot{x} < 0 \\ x(t) < 0 & \Longrightarrow & \dot{x} > 0 \end{cases}$
- Let $x(0) = a$, and compact set $W = \{x \in R ||x| \leq a\}$
- It is clear that the no solution can leave the set W.
- There is a unique solution for $t \geq 0$.

Continuous Dependence on Initial Condition and Parameters

- Consider mathematical model $\dot{x} = f(t, x)$
- We always interested to find solutions with continuous dependence on $t_0,\ x_0,\ f$
- Continuous dependence on t_0 is obvious from
  $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\ ds.$
- Let $x(t)$ be the solution starting at $x(t_0) = x_0$ and is defined over $[t_0, t_1]$.
- The solution depends continuously on x_0, if the solution starting nearby is defined over the same interval and remain nearby.

$$\text{Given } \epsilon > 0\ \exists \delta > 0 \text{ s.t. } \forall z_0 \in S = \{x \in R^n | \|x - x_0\| < \delta\}$$

the equation $\dot{x} = f(t, x)$ has a unique solution $z(t)$, defined over $[t_0, t_1]$, $z(t_0) = z_0$, and $\|z(t) - x(t)\| < \epsilon\ \ \forall t \in [t_0, t_1]$

Continuous Dependence on Parameters

- Let us consider changing parameters by perturbation on f
- f is continuously dependent on a set of parameters $\lambda \in R^p$, i.e. $f = f(t, x, \lambda)$.
- These parameters could represent physical parameters of system.
- Perturbation of these parameters accounts for modeling errors or changes in the parameters.
- Let $x(t, \lambda_0)$ be a solution of $\dot{x} = f(t, x, \lambda_0)$ defined over $[t_0, t_1]$ with $x(t_0, \lambda_0) = x_0$.
- Continuous dependence on λ if:

$$\text{Given } \epsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall \lambda \in \Lambda = \{\lambda \in R^p | \|\lambda - \lambda_0\| < \delta\}$$

the equation $\dot{x} = f(t, x, \lambda)$ has a unique solution $x(t, \lambda)$, defined over $[t_0, t_1]$, $x(t_0, \lambda) = x_0$, and $\|x(t, \lambda) - x(t, \lambda_0)\| < \epsilon \ \ \forall t \in [t_0, t_1]$

Continuous Dependence on Initial Conditions and Parameters

- Continuous dependence on initial conditions and parameters can be studied simultaneously.
- **Theorem 3.5:** *Let $f(t,x,\lambda)$ be continuous in t,x,λ and locally Lip. in x (uniformly in t and λ) on $[t_0,t_1]\times D\times\{\|\lambda-\lambda_0\|\le c\}$, where $D\subset R^n$. Let $x(t,\lambda_0)$ be a solution of $\dot{x}=f(t,x)$, $\quad x(t_0,\lambda_0)=x_0\in D$ (nominal solution). Suppose $x(t,\lambda_0)$ is defined and belongs to D $\forall t\in[t_0,t_1]$. Then,*

$$\text{Given } \epsilon>0\ \exists\delta>0 \text{ s.t. if} \|z_0-x_0\|<\delta \text{ and } \|\lambda-\lambda_0\|<\delta$$

then there is a unique solution $z(t,\lambda)$ of $\dot{x}=f(t,x,\lambda)$ defined on $[t_0,t_1]$ (solution for perturbed system), with $z(t_0,\lambda)=z_0$ and $z(t,\lambda)$ satisfies

$$\|z(t,\lambda)-x(t,\lambda_0)\|<\epsilon,\quad \forall\ t\in\ [t_0,t_1]$$

Sensitivity Analysis

- Suppose $f(t,x,\lambda)$ is continuous in (t,x,λ) and has continuous first partial derivatives w.r.t. x and λ $\forall$ $(t,x,\lambda) \in [t_0,t_1] \times R^n \times R^p$. Let λ_0 be a nominal value of λ and $x(t,\lambda_0)$ be the unique solution of the nominal state equation over $[t_0,t_1]$:
$$\dot{x} = f(t,x,\lambda_0) \qquad \text{with } x(t_0) = x_0$$
- From previous theorem, we see that for λ sufficiently close to λ_0, the sate equation
$$\dot{x} = f(t,x,\lambda) \qquad \text{with } x(t_0) = x_0$$
has a unique solution $x(t,\lambda)$ over $[t_0,t_1]$ that is close to the nominal solution $x(t,\lambda_0)$.
- Continuous differentiability of f w.r.t. x and λ implies differentiability of the solution $x(t,\lambda)$ w.r.t λ near λ_0.
$$x(t,\lambda) = x_0 + \int_{t_0}^{t} f(s,x(s,\lambda))\, ds$$

Sensitivity Analysis

- Taking partial derivative w.r.t. λ:

$$x_\lambda(t,\lambda) = \int_{t_0}^{t} \left[\frac{\partial f}{\partial x}(s, x(s,\lambda), \lambda) x_\lambda(s,\lambda) + \frac{\partial f}{\partial \lambda}(s, x(s,\lambda), \lambda)\right] \, ds$$

where $x_\lambda(t,\lambda) = \frac{\partial x(t,\lambda)}{\partial \lambda}$.

- Differentiating w.r.t. t:

$$\frac{\partial}{\partial t} x_\lambda(t,\lambda) = A(t,\lambda) x_\lambda(t,\lambda) + B(t,\lambda), \quad x_\lambda(t0,\lambda) = 0$$

where $A(t,\lambda) = \left.\frac{\partial f(t,x,\lambda)}{\partial x}\right|_{x=x(t,\lambda)}, \quad B(t,\lambda) = \left.\frac{\partial f(t,x,\lambda)}{\partial \lambda}\right|_{x=x(t,\lambda)}$

- For λ close to λ_0, $A(t,\lambda)$ and $B(t,\lambda)$ are defined on $[t_0, t_1]$. Hence, $x_\lambda(t,\lambda)$ is defined on the same interval.
- At $\lambda = \lambda_0$, the r.h.s. of the above equation depends only on the nominal solution $x(t,\lambda_0)$.

Sensitivity Equation:

- Let $S(t) = x_\lambda(t, \lambda_0)$. Then, $S(t)$ satisfies:

$$\dot{S}(t) = A(t, \lambda)S(t) + B(t, \lambda_0), \qquad S(t_0) = 0 \tag{5}$$

- The function $S(t)$ is called the *sensitivity function*
- Equ. (5) is called *sensitivity equation*
  - It provides first order estimate of the effect of parameter variations
  - It can also be used to approximate the solution when λ is close to λ_0.
- For small $\|\lambda - \lambda_0\|$, $x(t, \lambda)$, is expanded to a Taylor series

$$x(t, \lambda) \;\; = \;\; x(t, \lambda_0) + S(t)(\lambda - \lambda_0) + \text{high-order terms} \tag{6}$$

  - Neglect the higher order terms
  - Significance of (6): Knowing nominal solution and sensitivity function suffices for approximating the solution for all λ in a small ball centered at λ_0.

Procedure of calculating the sensitivity function $S(t)$

1. Solve the nominal state equation for the nominal solution $x(t, \lambda)$
2. Evaluate the Jacobian matrices

$$
\begin{aligned}
A(t, \lambda_0) &= \frac{\partial f(t, x, \lambda)}{\partial x}\Big|_{x=x(t,\lambda_0),\lambda=\lambda_0} \\
B(t, \lambda_0) &= \frac{\partial f(t, x, \lambda)}{\partial \lambda}\Big|_{x=x(t,\lambda_0),\lambda=\lambda_0}
\end{aligned}
\tag{7}
$$

3. Solve the sensitivity equation (5)

- Except for some trivial cases, these equations should be solved numerically

Alternative approach to calculate $S(t)$

- Solve the nominal solution and the sensitivity function simultaneously:
  - appending the variational equation (5) with original state equation
  - set $\lambda = \lambda_0$ to obtain $(n + np)$ augmented equation

$$\begin{aligned} \dot{x} &= f(t, x, \lambda_0), \quad x(t_0) = x_0 \\ \dot{S} &= \left[\frac{\partial f(t, x, \lambda)}{\partial x}\right]\bigg|_{\lambda=\lambda_0} S + \left[\frac{\partial f(t, x, \lambda)}{\partial \lambda}\right]\bigg|_{\lambda=\lambda_0}, \quad S(t_0) = 0 \end{aligned} \quad (8)$$

  which is solved numerically.
- if $f(t, x, \lambda) = f(x, \lambda) \rightsquigarrow$ (8) is autonomous as well

Example

- Consider
$$\begin{aligned} \dot{x}_1 &= x_2 = f_1(x_1, x_2) \\ \dot{x}_2 &= -c \sin x_1 - (a + b \cos x_1)x_2 = f_2(x_1, x_2) \end{aligned}$$
- nominal values $a_0 = 1, \; b_0 = 0, \; c_0 = 1.$
- nominal system
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - x_2 \end{aligned}$$
- Jacobian matrices:
$$\begin{aligned} \frac{\partial f}{\partial x} &= \begin{bmatrix} 0 & 1 \\ -c \cos x_1 + b x_2 sin x_1 & -(a + b \cos x_1) \end{bmatrix} \\ \frac{\partial f}{\partial \lambda} &= [\frac{\partial f}{\partial a} \; \frac{\partial f}{\partial b} \; \frac{\partial f}{\partial c}] = \begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -x_2 \cos x_1 & -sin x_1 \end{bmatrix} \end{aligned}$$

Example

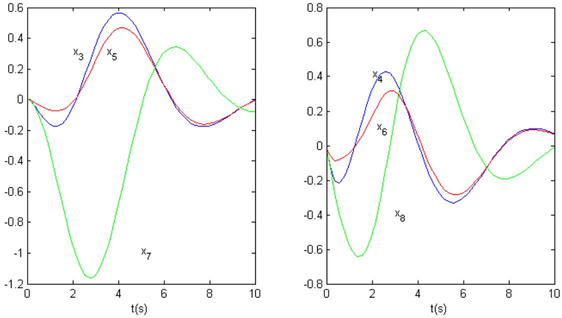
- Substitute the nominal values in Jacobian matrices and let

$$S = \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} & \frac{\partial x_1}{\partial c} \\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} & \frac{\partial x_2}{\partial c} \end{bmatrix} \Bigg|_{\text{nominal}}$$

- Then (8) is given by

$$\begin{aligned}
\dot{x}_1 &= x_2, & x_1(0) &= 1 \\
\dot{x}_2 &= -\sin x_1 - x_2, & x_2(0) &= 1 \\
\dot{x}_3 &= x_4, & x_3(0) &= 0 \\
\dot{x}_4 &= -x_3 \cos x_1 - x_4 - x_2, & x_4(0) &= 0 \\
\dot{x}_5 &= x_6, & x_5(0) &= 0 \\
\dot{x}_6 &= -x_5 \cos x_1 - x_6 - x_2 \cos x_1, & x_6(0) &= 0 \\
\dot{x}_7 &= x_8, & x_7(0) &= 0 \\
\dot{x}_8 &= -x_7 \cos x_1 - x_8 - \sin x_1, & x_8(0) &= 0
\end{aligned}$$

Example



Sensitivity function

- $x_3, \; x_5, \; x_7$ are sensitivity of x_1 with respect to $a, \; b, \; c$.
- $x_4, \; x_6, \; x_8$ are sensitivity of x_2 with respect to $a, \; b, \; c$.
- The solution is more sensitive to variations in c than a and b.
- This pattern is consistent for other initial states.

Comparison Principle

- Most often, it is interested to know the upper bound of $x(t)$ in state equation $\dot{x} = f(t,x)$ without computing the solution itself.
- The Gronwall-Bellman inequality can provide an upper bound for $x(t)$:
- **Gronwall-Bellman inequality Lemma**:*Let $\lambda : [a,b] \rightarrow \mathcal{R}$ be continuous and $\mu : [a,b] \rightarrow \mathcal{R}$ be continuous and nonnegative. If a continuous function $y : [a,b] \rightarrow \mathcal{R}$ satisfies*

$$y(t) \leq \lambda(t) + \int_a^t \mu(s) y(s) ds \quad \text{for } a \leq t \leq b$$

then for t on the same interval

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp\left[\int_a^t \mu(\tau) d\tau\right] ds$$

In particular if $\lambda(t) \equiv \lambda$ is a constant, then $y(t) \leq \lambda \exp[\int_a^t \mu(\tau) d\tau]$
If, in addition, $\mu(t) \equiv \mu \geq 0$ is a constant, then $y(t) \leq \lambda \exp[\mu(t-a)]$

Comparison Principle

- Comparison lemma is another tool for finding upper bound on solution
- The comparison lemma compares the solution of the differential inequality $\dot{v}(t) \leq f(t, v(t))$ with the solution of the differential equation $\dot{u} = f(t, u(t))$.
- $v(t)$ is a scalar differentiable function named a solution of the differential inequality.
- This lemma is also applicable when $v(t)$ is not differentiable, but has an upper right-hand derivative $D^+v(t) = \limsup_{h\to 0^+} \frac{v(h+t)-v(t)}{h}$.
  - if $v(t)$ is differentiable at $t \rightsquigarrow D^+v(t) = \dot{v}(t)$
  - If $\frac{1}{h}[v(h+t) - v(t)] \geq g(t,h) \quad \forall h \in (0, b]$ and $\lim_{h\to 0^+} g(t,h) = g_0(t)$ the $D^+v(t) \geq g_0(t)$

Comparison Lemma

- *Consider the scalar differential equation*

$$\dot{u} = f(t, u) \quad u(t_0) = u_0$$

where $f(t, u)$ is continuous in t and locally Lipschitz in u, for all $t \geq 0$ and all $u \in J \subset \mathcal{R}$. Let $[to, T)$ (T could be ∞) be the maximal interval of existence of the solution $u(t)$. Let $v(t)$ be a continuous function whose upper right-hand derivative $D^+v(t)$ satisfies the differential inequality

$$D^+v(t) \leq f(t, v) \quad v(t_0) \leq u_0$$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \leq u(t)$ for all $t \in [t_0, T)$.

- prove it

Example

$$\dot{x} = f(x) = -(1+x^2)x, \quad x(0) = a$$

- $f(x)$ is locally Lipshitz $\rightsquigarrow$ x has unique solution in $[0, t_1)$
- let $v(t) = x^2$, $v(t)$ is differentiable

$$\begin{aligned} \dot{v}(t) &= 2x(t)\dot{x}(t) = -2x^2(t) - 2x^4(t) \leq -2x^2(t) \\ \therefore \dot{v}(t) &\leq -2v(t) \quad v(0) = a^2 \end{aligned}$$

- Now define $u(t)$ as

$$\dot{u}(t) = -2u(t) \quad u(0) = a^2 \rightarrow u(t) = a^2 e^{-2t}$$

- Therefore, using comparison lemma yields

$$x(t) = \sqrt{(v(t)} \leq e^{-t}|a|$$