Signals and Systems
Lecture 2: LTI Systems

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DT Convolution

- Using sampling property of $\delta[.]$ yields
  - $x[n] = x[0]\delta[n] + x[1]\delta[n-1] + \ldots = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

DT Convolution

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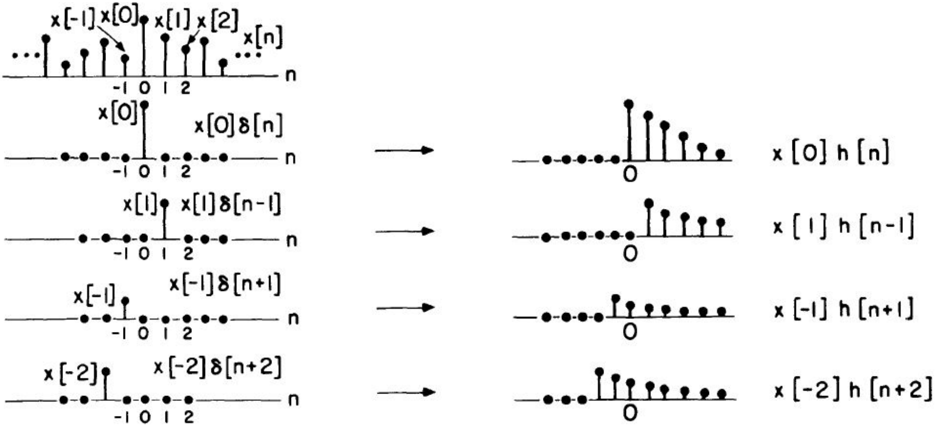
  - $x[n] = x[0]\delta[n] + x[1]\delta[n-1] + \ldots = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- Then, for LTI system
  - Consider $h[n]$ be impulse response (i.e., $x[n] = \delta[n] \rightsquigarrow y[n] = h[n]$)
  - $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$

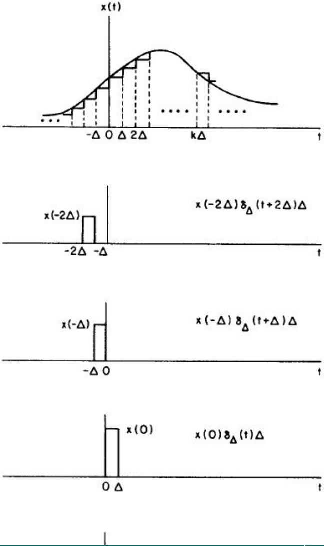
DT Convolution



CT Convolution

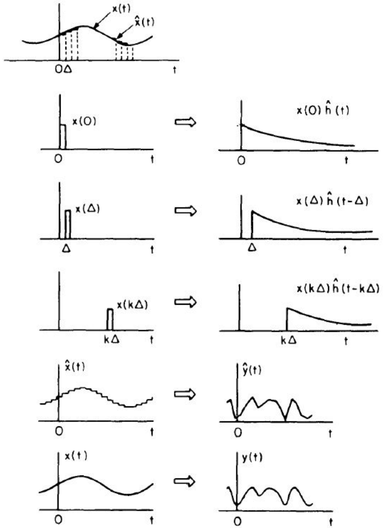
$$x(t) \cong x(0)\delta_\Delta(t)\Delta + x(\Delta)\delta_\Delta(t-\Delta)\Delta + x(-\Delta)\delta_\Delta(t+\Delta)\Delta + \dots$$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \qquad (1)$$



CT Convolution

- $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$
- $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$



CT Convolution

- $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$
- To calculate the convolution:
  1. Find $h(-\tau)$
  2. Shift it by t: $h(t-\tau)$
  3. Multiply it by $x(\tau)$: $x(\tau)h(t-\tau)$
  4. Calculate $\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$
- Convolution properties
  - $x[n] * \delta[n] = x[n]$
  - $x[n] * \delta[n-n_0] = x[n-n_0]$

LTI systems Properties

- Commutative Property
  - $\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$
  - $\sum_{k=\infty}^{\infty} x[k]h[n-k] = x[n] * h[n] = h[n] * x[n] = \sum_{r=\infty}^{\infty} x[n-r]h[r]$
- Distributive Property
  - $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$
  - $x[n] * [h_1[n] + h_2[n]] = x[n] * h_1[n] + x[n] * h_2[n]$
- Associative Property
  - $x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$
  - $x[n] * [h_1[n] * h_2[n]] = [x[n] * h_1[n]] * h_2[n]$
  - The impulse response of two LTI systems does not depend of the order they are cascaded

LTI systems Properties

- In LTI systems impulse response provides unique relation between input and output signals
  - In nonlinear systems impulse response provides same information as other input responses
  - Example: If impulse response of a system is $h[n] = u[n] - u[n-3]$ then
    - For linear system $\rightsquigarrow$ $y[n] = x[n] + x[n-1] + x[n-2]$
    - For nonlinear systems several system can be defined to have them same impulse response $h[n]$:
      $y[n] = (x[n] + x[n-1] + x[n-2])^m \underbrace{\longrightarrow}_{m=2} y[n] = (\delta[n] + \delta[n-1] + \delta[n-2])^2 =$
      $\delta^2[n] + \delta^2[n-1] + \delta^2[n-2] + 2\delta[n]\delta[n-1] + 2\delta[n]\delta[n-2] + 2\delta[n-1]\delta[n-2] =$
      $u[n] - u[n-3]$
- Commutative property is specified for linear systems:
  - Example: Consider the cascade systems:
    - The first takes square root and the second one square then for input -2 output is -2
    - The first takes square and the second one square root then for input -2 output is 2

LTI systems Properties

- In LTI systems impulse response represents a unique relation of I/O $\rightsquigarrow$ properties of the system can be defined based on its impulse response
- With Memory/ Memoryless
  - at any time Output depends only on input at the same time
  - By considering convolution definition $\rightsquigarrow$ for $n \neq 0, h[n] = 0$ (for $t \neq 0, h(t) = 0$)
  - $\therefore h[n] = k\delta[n] (h(t) = k\delta(t))$
- Invertibility
  - If the invert of a system exists then by cascading the system and its inverse the output equals to the input.
  - $h_1(t) * h_2(t) = \delta(t) (h_1[n] * h_2[n] = \delta[n])$

LTI systems Properties

- Causality
  - Output depends on present and past values of input
  - Based on convolution definition $y[n]$ should not depend on $x[k]$ $\ for\ k > n$
  - $\therefore h[n-k] = 0$ for $n < k \rightsquigarrow h[n] = 0\ for\ n < 0 (h(t) = 0\ for\ t < 0$
  - The impulse response should be causal.
  - For linear systems: causality $\equiv$ initial rest (if the input is 0 up to some time, output is also zero at up to that time)
  - Can causality be verified by impulse response of nonlinear time invariant or linear time varying systems?

- Stability
  - Bounded input $\rightsquigarrow$ bounded output
  - If $|x(t)| < B : |y(t)| = |\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau| \leq \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|d\tau \leq B\int_{-\infty}^{\infty} |h(\tau)|d\tau$
  - To have bounded output in CT, impulse response should be absolutely integrable: $\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$
  - To have bounded output in DT, impulse response should be absolutely summable: $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$
  - what we found id sufficient condition for stability, show that this is also necessary condition
- If impulse response of an LTI system is periodic, is it causal? Is it stable?

Step Response

- Sometimes step response is used to study an LTI system
- step response in DT: $s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[k]$, $h[n] = s[n] - s[n-1]$
- step response in CT: $s(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau)d\tau, \quad h(t) = \frac{ds(t)}{dt}$
- Causality: $s(t) = 0$ for $t < 0$
- Invertibility: $u[n] * h_1[n] * h_2[n] * u[n] = u[n] * u[n] = r[n]u[n]$, $r[n]$: ramp
- Memoryless: $s[n] = u[n] * k\delta[n] = ku[n]$
- Stability: $\sum_{-\infty}^{\infty} |s[n] - s[n-1]| < \infty$ (Exercise: find stability condition for CT?)
- Example: $y = \int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = x(t) * u(t)$
- Exercise: Find definition of convolution and causality condition for linear time-varying systems

Singularity Functions

- Impulse response definition based on convolution: $x(t) = x(t) * \delta(t)$

Singularity Functions

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- If $x(t) = 1 \rightsquigarrow 1 = x(t) * \delta(t) = \int_{-\infty}^{\infty} \delta(\tau) x(t-\tau) d\tau \rightsquigarrow \int_{-\infty}^{\infty} \delta(\tau) d\tau = 1$ unit impulse has unit area

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- $g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau - t)\delta(\tau) d\tau$

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- $g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau - t)\delta(\tau)d\tau$
  - $t = 0 \rightsquigarrow g(0) = \int_{-\infty}^{\infty} g(\tau)\delta(\tau)d\tau$ (another definition of unit impulse)
  - Now multiply the above equation by $f(0)$ $\rightsquigarrow g(0)f(0) = \int_{-\infty}^{\infty} g(\tau)f(0)\delta(\tau)d\tau$
  - substitute $g(-t)$ by $f(-t)g(-t) \rightsquigarrow g(0)f(0) = \int_{-\infty}^{\infty} g(\tau)f(\tau)\delta(\tau)d\tau$
  - $\therefore f(t)\delta(t) = f(0)\delta(t)$

Unit Doublet signals

- Consider a system that takes derivative of input signal: $y(t) = \frac{dx(t)}{dt}$
- Impulse response of such system is called unit doublet $u_1(t)$:
$\frac{dx(t)}{dt} = x(t) * u_1(t)$
- $\frac{d^2x(t)}{dt^2} = x(t) * u_2(t) = \frac{d}{dt}\frac{dx(t)}{dt} = x(t) * u_1(t) * u_1(t) \rightsquigarrow u_2(t) = u_1(t) * u_1(t)$
- $u_k = \underbrace{u_1 * u_1 * \ldots * u_1}_{ktimes}$
- $x(t) = 1 \rightsquigarrow 0 = \frac{dx(t)}{dt} = u_1(t) * x(t) = \int_{-\infty}^{\infty} u_1(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} u_1(\tau)d\tau$
- $\int_{-\infty}^{\infty} u_1(\tau)d\tau = 0$ (unit doublet has zero area)

- Consider a system that takes integral from input signal: $y(t) = \int_{-\infty}^{\infty} x(\tau)d\tau$
- $x(t) = \delta(t) \rightsquigarrow u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$
- $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau$
- By cascading two systems: $u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau)d\tau = tu(t)$ This is a unit ramp function
- $u_{-k} = \underbrace{u(t) * u(t) * ... * u(t)}_{ktimes} = \int_{-\infty}^{t} u_{k-1}(\tau)d\tau$
- $u_{-k}(t) = \frac{t^{k-1}}{(k-1)!}u(t)$
- One can define: $\delta(t) = u_0(t)$ and $u(t) = u_{-1}(t)$
- $u_k * u_r = u_{k+r}$ where (r and k can be positive or negative integers).