Nonlinear Control
Lecture 3: Fundamental Properties

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- The norm $\|x\|$ of a vector x is a real-valued function s.t.
  1. $\|x\| \geq 0 \ \forall x \in R^n, \quad \|x\| = 0$ iff $x = 0$
  2. $\|x+y\| \leq \|x\| + \|y\|, \ \forall x, y \in R^n$
  3. $\|ax\| = |a|\|x\| \ \ \forall a \in R, \ \forall x \in R^n$
- The class $p-norm, \ p \in [1, \infty)$ are defined by

$$\|x\|_p = (|x_1|^p + \ldots + |x_n|^p)^{1/p}$$

- $\|x\|_\infty = \max_i |x_i|$
- The three most common norms are:
  $\|x\|_1, \ \|x\|_\infty$, and the Euclidean norm $\|x\|_2 = \left(x^T x\right)^{1/2}$
- All p-norms are equivalent in the sense that $\exists c_1$ & c_2 s.t.:

$$c_1\|x\|_\alpha \leq \|x\|_\beta \leq c_2\|x\|_\alpha \ \ \forall x \in R^n$$

e.g.: $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$
$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$
$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$

- An $m \times n$ matrix A defines a linear mapping $y = Ax$ from R^n into R^m. The induced $p-norm$ of A is defined by:

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\|x\| \leq 1} \|Ax\|_p = \sup_{\|x\|=1} \|Ax\|_p$$

- for $p = 1, 2, \infty$, we have
  $\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|$
  $\|A\|_2 = \sigma_{max}(A) = \left[\lambda_{max}(A^T A)\right]^{1/2}$
  $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$
- we have
  $\|A\|_2 \leq \sqrt{\|A\|_1 \|A\|_\infty}$
  $\frac{1}{n}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty$
  $\frac{1}{m}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\|A\|_1$
  $\|AB\|_p \leq \|A\|_p \|B\|_p$
- **Hölder inequality:**
  $|x^T y| \leq \|x\|_p \|y\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad x, y \in R^n$

Set

- A set S is closed iff every convergent sequence $\{x_d\}$ with elements in S converges to a point in S.
- A set S is bounded if there is $r > 0$ s.t. $\|x\| \leq r$ for all $x \in S$.
- A set S is compact if it is closed and bounded.
- A set S is convex: if for every $x, y \in S$ and every real number $\theta,\ 0 < \theta < 1$, the point $\theta x + (1 - \theta)y \in S$.
  - In Euclidean space, an object is convex if for every pair of points within the object, every point on the straight line segment that joins them is also within the object.

Set

- A set S is closed iff every convergent sequence $\{x_d\}$ with elements in S converges to a point in S.
  - Convergence: A sequence $\{x_d\} \in S$, a normed linear space, converges to x, if $\|x_d - x\| \to 0$ as $d \to \infty$
- A set S is bounded if there is $r > 0$ s.t. $\|x\| \leq r$ for all $x \in S$.
- A set S is compact if it is closed and bounded.
- A set S is convex: if for every $x, y \in S$ and every real number θ, $0 < \theta < 1$, the point $\theta x + (1 - \theta) y \in S$.
  - In Euclidean space, an object is convex if for every pair of points within the object, every point on the straight line segment that joins them is also within the object.

Continuous Function

- A function f mapping a set S_1 into a set S_2 is denoted by $f : S_1 \rightarrow S_2$.
- f is continuous at x if, given $\epsilon > 0$, there is $\delta > 0$ s.t

$$\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \epsilon \tag{1}$$

- A function f is continuous on set S if it is continuous at every point of S
- A function f is uniformly continuous on S if given $\epsilon > 0$ there is $\delta > 0$ (dependent only on ϵ,not the point in the domain) s.t. (1) holds for all $x, y \in S$.
- For uniform continuity, the same constant δ works for all points in the set.
- f is uniformly continuous on a set $S \Rightarrow$ it is continuous on S. But the opposite is not true in general.
- If S is a compact set, then continuity $\equiv$ uniform continuity.

Continuous Differentiable Function

- A function $f : R \longrightarrow R$ is differentiable at x if

$$\acute{f}(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- A function $f : R^n \longrightarrow R^m$ is continuously differentiable at a point x_0 if $\frac{\partial f_i}{\partial x_j}$ exist and are continuous at x_0 for $1 \leq i \leq m,\ 1 \leq j \leq n$.
- A function f is continuously differentiable on a set S if it is continuously differentiable at every point of S.

Existence

- This section provides sufficient condition for uniqueness and existence solution of the initial value problem

$$\dot{x} = f(t,x), \quad x(t_0) = x_0 \tag{2}$$

- Existence of solution is provided by continuity
- A solution of (2) over an interval $[t_0, t_1]$:
  $x : [t_0, t_1] \longrightarrow R^n$ s.t. $\dot{x}(t)$ is defined, $\dot{x}(t) = f(t, x(t))\ \forall t \in [t_0, t_1]$
  - If f is continuous in t and $x \rightsquigarrow$ the solution $x(t)$ is continuously differentiable.
  - Assume f is continuous in x but only *piecewise continuous* in $t \rightsquigarrow x(t)$ is *only piecewise continuously differentiable*.
- This allows time-varying input with step changes in time.

Existence

- A differential equation might have many solutions, e.g.

$$\dot{x} = x^{1/3}, \quad x(0) = 0 \tag{3}$$

- $x(t) = (2t/3)^{3/2}$ and $x(t) = 0$ $\rightsquigarrow$ the solution is not unique.
- $\therefore f$ is continuous $\rightsquigarrow$ continuity is not sufficient to ensure uniqueness.
- Continuity of f guarantees *at least* one solution.

Existence and Uniqueness

- **Theorem 3.1 (Lipschitz condition: Local Existence and Uniqueness)**
  Let $f(t,x)$ be piecewise continuous in t and satisfy the Lipschitz condition:

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\| \quad \forall x,y \in B = \{x \in R^n | \|x - x_0\| \leq r\}, \quad \forall t \in [t_0, t_1]$$

  Then, there exists $\delta > 0$ such that the state equation $\dot{x} = f(t,x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.
  - The function f satisfying Lipschitz condition is called Lipschitz in x
  - The constant L is called the Lipschitz constant.
  - A function can be locally or globally Lipschitz.

Existence and Uniqueness

- A function $f(x)$ is said to be **locally Lipschitz** on a domain $D \subset R^n$ (open and connected set) if each point of D has a neighborhood D_0 such that $f(x)$ satisfies the Lipschitz condition for all points on D_0 with **some** Lipschitz constant L_0.
- A function $f(x)$ is Lipschitz on a set W if it satisfies Lipschitz condition for all points with the **same** Lipschitz constant.
- $\therefore$ A locally Lipschitz function on D is not necessarily Lipschitz on D since the Lipschitz condition may not hold uniformly (with the same Lipschitz constant) for all points in D.
- A function $f(x)$ is said to be **globally Lipschitz** if it is Lipschitz on R^n.
- The same terminology holds for $f(t, x)$ if the Lipschitz condition is hold uniformly in t for all t in a certain interval.

Existence and Uniqueness

- A function $f(t,x)$ is said to be locally Lipschitz on $[a,b] \times D \subset R \times R^n$ if each point of $x \in D$ has a neighborhood D_0 such that $f(t,x)$ satisfies the Lipschitz condition for **same** Lipschitz constant L_0 on $[a,b] \times D_0$.
  - If it is true for $\forall [a,b] \subset [t_0, \infty] \Longrightarrow f$ is locally Lipschitz on $[t_0, \infty] \times D$.
- If f is scalar, $f : R \longrightarrow R$, the Lipschitz condition can be expressed as:

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L$$

- The line connecting every two points of f, cannot have a slope $> L$.
- $\therefore$ If a function has infinite slope at some points, the function cannot be locally Lipschitz at those points.
- Discontinuous functions cannot be locally Lipschitz at the points of discontinuity.

Existence and Uniqueness

- **Example:** $f(x) = x^{1/3}$ is not locally Lip. at $x = 0$ since $f'(x) = (1/3)x^{-2/3} \longrightarrow \infty$ as $x \longrightarrow 0$.
- If $f'(x)$ in some region is bounded by k, then f is lip on that region with Lip. constant $L = k$.
- This fact is also true for vector valued functions
- **Lemma 3.1**: *Let $f : [a, b] \times D \longrightarrow R^m$ be continuous for some domain $D \in R^n$. If for a convex subset $W \subset D$ there is a constant $L \geq 0$ s.t.*

$$\|\frac{\partial f}{\partial x}(t, x)\| \leq L \qquad \text{on } [a, b] \times W,$$

then $\|f(t, x) - f(t, y)\| \leq L\|x - y\|$ for all $t \in [a, b]$, $x \in W$, and $y \in W$.
- $\therefore$ a Lipschitz constant can be calculated using $\partial f / \partial x$

Proof of Lemma 3.1

- Let $\|.\|_p$ be any norm $p \in [1, \infty]$ and determine q s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Fix t on $[a, b]$ and assume $x \in W$, $y \in W$.
- Define $\gamma(s) = (1-s)x + sy, \quad s \in R, \quad \gamma(s) \in D,$
- $W \subset D$ is convex $\rightsquigarrow$ $\gamma(s) \in W$ for $0 \leq s \leq 1$.
- Take $z \in R^m$ s.t.

$$\|z\|_q = 1, \quad z^T[f(t,y) - f(t,x)] = \|f(t,y) - f(t,x)\|_p$$

- set $g(s) = z^T f(t, \gamma(s))$. Since, $g(s)$ is a continuously differentiable real-valued function over the open interval which includes $[0, 1]$, from mean-value theorem , $\exists s_1 \in (0, 1)$ s.t.

$$g(1) - g(0) = g'(s_1)$$

Proof of Lemma 3.1 Cont'd

- Evaluating g at $s = 0$ and $s = 1$:

$$z^T[f(t,y) - f(t,x)] \;=\; z^T \frac{\partial f}{\partial x}(t,\gamma(s_1))(y-x)$$

- and using chain rule in calculating $g'(s)$ and Hölder inequality, $|z^T w| \leq \|z\|_q \|w\|_p$:

$$\|f(t,y) - f(t,x)\|_p \;\leq\; \|z\|_q \left\| \frac{\partial f}{\partial x}(t,\gamma(s_1)) \right\|_p \|y-x\|_p \leq L\,\|y-x\|_p$$

Existence and Uniqueness

- If f is Lip. on W, $\Rightarrow$ it is uniformly continuous on W, (prove it) but the converse is not true
- The function $f(x) = x^{1/3}$ is continuous on R, but it's not locally lip on $x = 0$.
- Lip. condition is weaker than continuous differentiability condition :

Existence and Uniqueness

- **Lemma 3.2** *If $f(t,x)$ and $[\frac{\partial f}{\partial x}](t,x)$ are continuous on $[a,b] \times D$ for some domain $D \subset R^n$, then f is locally Lip. in x on $[a,b] \times D$.*
- **Proof:**
  - For $x_0 \in D$, let r be so small that the ball $D_0 = \{x \in R^n | \|x - x_0\| \leq r\}$ is contained in D
  - The set D_0 is convex and compact
  - By continuity, $\frac{\partial f}{\partial x}$ is bounded on $[a,b] \times D_0$.
  - Let L_0 is a bound for $\frac{\partial f}{\partial x}$ on $[a,b] \times D_0$
  - By Lemma 3.1, $f(t,x)$ is Lip. on $[a,b] \times D_0$ with Lip. constant L_0.
- **Lemma 3.3**: *If $f(t,x)$ and $[\frac{\partial f}{\partial x}](t,x)$ are continuous on $[a,b] \times R^n$, then f is globally Lip. in x on $[a,b] \times R^n$ iff $[\frac{\partial f}{\partial x}]$ is uniformly bounded on $[a,b] \times R^n$.*
  - $x(t)$ is uniformly bounded if $\exists c > 0$, independent of $t_0 > 0$, and for every $a \in (0,c)$, there is $\beta = \beta(a) > 0$, independent of t_0, s.t.

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta, \forall t \geq t_0$$

Example 1

$$f(x) = \begin{bmatrix} -x_1 + x_1x_2 \\ x_2 - x_1x_2 \end{bmatrix}$$

- f is continuously differentiable on $R^2 \Longrightarrow f$ is locally Lip. on R^2.
- f is not globally Lip. since $\frac{\partial f}{\partial x}$ is not uniformly bounded on R^2.
- However, it is Lip. on any compact set on R^2.
- Find the Lip. constant on set $W = \{x \in R^2 | |x_1| \leq a_1, \ |x_2| \leq a_2\}$.
  - fist find jacobian matrix $\frac{\partial f}{\partial x} = \begin{bmatrix} -1 + x_2 & x_1 \\ -x_2 & 1 - x_1 \end{bmatrix}$
  - Use ∞ norm for vectors and induced norm for matrices:

$$\begin{aligned} \|\frac{\partial f}{\partial x}\|_\infty &= max\{|-1+x_2| + |x_1|, |x_2| + |1-x_1|\} \\ |-1+x_2| + |x_1| &\leq 1 + a_2 + a_1, \quad |x_2| + |1-x_1| \leq a_2 + 1 + a_1 \\ \|\frac{\partial f}{\partial x}\|_\infty &\leq 1 + a_1 + a_2 \rightsquigarrow L_0 = 1 + a_1 + a_2 \end{aligned}$$

Example 2

$$f(x) = \begin{bmatrix} x_2 \\ -sat(x_1 + x_2) \end{bmatrix}$$

- f is **not** continuously differentiable on R^2.
- Lip. condition is evaluated by definition.
- Use $\|.\|_2$ and also note that

$$\begin{aligned} |sat(\eta) - sat(\zeta)| &\leq |\eta - \zeta| \\ \therefore \|f(x) - f(y)\|_2 &\leq (x_2 - y_2)^2 + (x_1 + x_2 - y_1 - y_2)^2 \\ &= (x_1 - y_1)^2 + 2(x_1 - y_1)(x_2 - y_2) + 2(x_2 - y_2)^2 \end{aligned}$$

- We have
$$a^2 + 2ab + 2b^2 = \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \leq \lambda_{max} \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right\} \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|_2^2$$
- $\therefore \|f(x) - f(y)\|_2 \leq \sqrt{2.618}\|x - y\|_2, \quad \forall x, y \in R^2$

- If we use the more conservative inequality

$$a^2 + 2ab + 2b^2 \leq 2a^2 + 3b^2 \leq 3(a^2 + b^2)$$

- The Lip constant $\sqrt{3}$ is obtained.
- **Therefore**
  - Type of norm does not affect the Lip. property, but it does affect the Lip. constant
  - If the Lip. condition is satisfied for some L_0, it is also hold for all $L > L_0$.
  - Lip. constant is not unique
  - Theorem 3.1 is a local theorem
  - It guarantees the existence and uniqueness for the interval $[t_0, t_0 + \delta]$.
  - Existence and uniqueness for the interval $[t_0, t_1]$ is not clear.

- In general, we cannot extend δ s.t. $t+\delta = t_1$
  - $\therefore$ there is a maximum interval $[t_0, T]$ that the unique solution which starts from (t_0, x_0) exists.
  - T might be smaller than t_1, in this case when $t \longrightarrow T$, the solution leaves the set on which f is locally Lip.
- **Example 3.3** $\dot{x} = -x^2, \quad x(0) = -1$
- f is locally Lip. for all $x \in R$.
- It is locally Lip. on all compact subset of R

$$x(t) = \frac{1}{t-1} \qquad \text{Unique solution} \quad \text{on } [0,1]$$

- As $t \longrightarrow 1$ $x(t)$ leaves the set.
- Finite escape time indicates that the trajectories go to infinity in finite time.
- $\therefore$ The trajectory has finite escape time at $t = 1$

When the solution interval can be extended indefinitely?

- One way to keep the solution $x(t)$ always in the set: $f(t,x)$ be glob. Lip.
- **Theorem 3.2 (Global Existence and Uniqueness)** *Suppose that $f(t,x)$ is piecewise continuous in t and satisfies*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\| \quad \forall x, y \in R^n, \; \forall t \in [t_0, t_1]$$

Then, $\dot{x} = f(t,x), \;\; x(t_0) = x_0$ has a unique solution on $[t_0, t_1]$.

- **Example 3.4:** $\dot{x} = A(t)x + g(t) = f(t,x)$
  - where $A(t)$ and $g(t)$ are piecewise continuous functions in t.
  - Over any finite interval, elements of $A(t)$ and $g(t)$ are bounded

$$\|A(t)\| \leq a \;\; \text{using any} \;\; \text{induced} \;\; \text{norm}$$

  - All conditions of Theorem 3.2 is satisfied since $\forall x, y \in R^n$ and $t \in [t_0, t_1]$:

$$\|f(t,x) - f(t,y)\| = \|A(t)(x - y)\| \leq \|A(t)\|\|x - y\| \leq a\|x - y\|$$

- Example 3.4. Contd.
  - Linear System has a unique solution over $[t_0, t_1]$.
  - t_1 can be arbitrarily large $\rightsquigarrow$ if $A(t)$ and $g(t)$ are piecewise continuous functions, system has a unique solution for $t \geq t_0$ and cannot have "finite escape time".
- The global Lip. condition is reasonable for linear systems.
- In general, it is rarely satisfied for nonlinear systems
- Local Lip. condition is essentially related to smoothness of f
- It is automatically satisfied if f is continuously differentiable
- Except for hard nonlinearities which are idealization of nonlinear phenomena, physical system models satisfy Lip. condition
- Continuous functions which are not locally Lip. are rare in practice.
- However, the global Lip. condition cannot be satisfied by many physical systems.

- Theorem 3.2 provides conservative condition on unique solution of nonlinear systems
- **Example 3.5:** $\dot{x} = -x^3 = f(x)$
  - $f(x)$ is not globally Lip. since Jacobian $\frac{\partial f}{\partial x}$ is not bounded in R.
  - However, for $x(t_0) = x_0$, the unique solution is given by

$$x(t) = sign(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2(t - t_0)}}$$

- By having some knowledge about the solution $x(t)$, one can prove less conservative condition for uniqueness using local Lip. condition on f

- **Theorem 3.3:** *Let $f(t,x)$ is piecewise continuous in t and is locally Lip. in x for all $t \geq t_0$ and all $x \in D \subset R^n$. Let W be a compact subset of D, $x_0 \in W$ and every solution of $\dot{x} = f(t,x), \quad x(t_0) = x_0$ lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$.*
- **Proof:**
  - The proof is based on the fact that if the solution remains in the set W, it cannot have "finite escape time".
  - By Theorem 3.1, the unique solution exist in the interval $[t_0, t_0 + \delta]$. From the previous discussion we know that if T is finite, the solution must leave D, however, since the solution never leaves W, we conclude that $T = \infty$.
- The problem in applying this theorem is to show that the solution never leaves the set W.
- We desire to check the assumption that every solution lies in a compact set without actually solving the differential equation.
- Lyapunov's stability theorem is an important tool for this purpose.

Example 3.6:

$$\dot{x} = -x^3 = f(x)$$

- $f(x)$ is locally Lip. on R
- $\begin{cases} x(t) > 0 \implies \dot{x} < 0 \\ x(t) < 0 \implies \dot{x} > 0 \end{cases}$
- Let $x(0) = a$, and compact set $W = \{x \in R || x| \leq a\}$
- It is clear that no solution can leave the set W.
- There is a unique solution for $t \geq 0$.

Summery

- Solution exitance for $\dot{x} = f(x,t)$ is achieved by continuity or at least piecewise continuity of function f in t.
- Lipschitz condition can provide sufficient condition for unique solution
- Theorem 3.1: *Let $f(t,x)$ be piecewise continuous in t and satisfy the Lipschitz condition:*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\| \quad \forall x, y \in B = \{x \in R^n | \|x - x_0\| \leq r\}, \quad \forall t \in [t_0, t_1]$$

Then, there exists $\delta > 0$ such that the state equation $\dot{x} = f(t,x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

Summery

- Locally Lipschitz
  - The condition is satisfied on a subset $D \subset R^n$
  - It guarantees unique solution over $[t_0, t_0 + \delta]$
  - A function $f(x)$ is Lipschitz on a set W if it satisfies Lipschitz condition for all points with the **same** Lipschitz constant.
  - To check the Lipschitz conation a convex subset $W \subset D$, it is sufficient to satisfy: $\|\frac{\partial f}{\partial x}(t,x)\| \leq L \quad \text{on } [a,b] \times W$.
  - To find Lip. constant, L, type of norm does not affect the Lip. property, but it does affect the Lip. constant.
  - Lip. constant is not unique.
  - Continuously differentiability of $f(t,x)$ on $[a,b] \times D$ guarantees f to be locally Lip.

Summery

- Globally Lipschitz
  - The condition is satisfied on R^n
  - It guarantees unique solution over $[t_0, t_1]$, (no matter how large t_1 is)
  - Continuously differentiability of $f(t, x)$**+** uniformly boundedness of $\frac{\partial f}{\partial x}$ on $[a, b] \times R^n$ guarantees f to be globally Lip.
  - uniformly boundedness of $\frac{\partial f}{\partial x}$ is a killer condition and difficult to be satisfied for nonlinear systems in practice.
  - By finding a compact subset W in which every solution of $\dot{x}$ lies entirely, locally Lip. also guarantees a unique solution for all $t \geq t_0$.