# Computational Intelligence
# Lecture 3: Simple Neural Networks for Pattern Classification

**Farzaneh Abdollahi**

Department of Electrical Engineering

Amirkabir University of Technology

Fall 2011

## Neural Processing

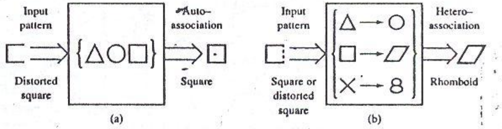
- Classification
- Discriminant Functions

## Learning

- Perceptron Learning Rule

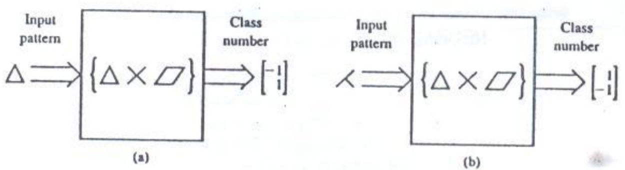
# Neural Processing

- One of the most applications of NN is in mapping inputs to the corresponding outputs $o = f(wx)$
- The process of finding $o$ for a given $x$ is named **recall**.
- Assume that a set of patterns can be stored in the network.
- **Autoassociation:** The network presented with a pattern similar to a member of the stored set, it associates the input with the closest stored pattern.
  - A degraded input pattern serves as a cue for retrieval of its original
- **Hetroassociation:** The associations between pairs of patterns are stored.




Association response: (a) autoassociation and (b) heteroassociation.

- **Classification:** The set of input patterns is divided into a number of classes. The classifier recall the information regarding class membership of the input pattern. The outputs are usually binary.
  - Classification can be considered as a special class of hetroassociation.
- *Recognition:* If the desired response is numbers but input pattern does not fit any pattern.

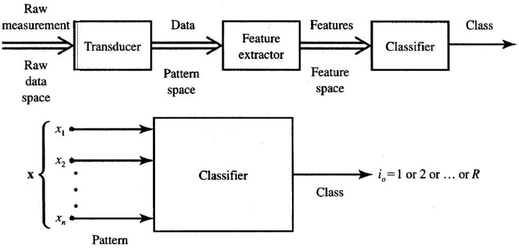



Classification response: (a) classification and (b) recognition.

- **Function Approximation:** Having I/O of a system, their corresponding function $f$ is approximated.
  - This application is useful for control
- In all mentioned aspects of neural processing, it is assumed the data is already stored to be recalled
- Data are stored in a network in learning process

# Pattern Classification

- The goal of pattern classification is to assign a physical object, event, or phenomenon to one of predefined classes.
- Pattern is quantified description of the physical object or event.
- Pattern can be based on time (sensors output signals, acoustic signals) or place (pictures, fingertips):
- Example of classifiers: disease diagnosis, fingertip identification, radar and signal detection, speech recognition
- Fig. shows the block diagram of pattern recognition and classification

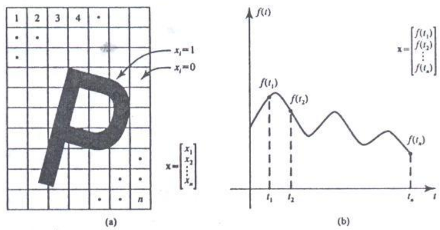


# Pattern Classification

- Input of feature extractors are sets of data vectors belonging to a certain category.
- Feature extractor compress the dimensionality as much as does not ruin the probability of correct classification
- Any pattern is represented as a point in $n$-dimensional Euclidean space $E^n$, called pattern space.
- The points in the space are $n$-tuple vectors $X = [x_1 \ \ldots \ x_n]^T$.
- A pattern classifier maps sets of points in $E^n$ space into one of the numbers $i_0 = 1, \ldots, R$ based on decision function $i_0 = i_0(x)$.
- The set containing patterns of classes 1,..., $R$ are denoted by $\S_i, \ldots \S_n$

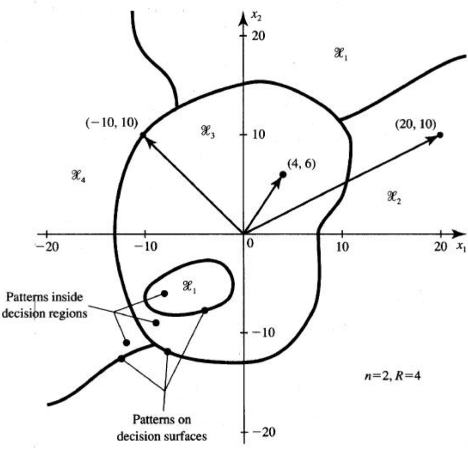
# Pattern Classification

- The fig. depicts two simple method to generate the pattern vector
  - Fig. a: $x_i$ of vector $X = [x_1 \; ... \; x_n]^T$ is 1 if $i$th cell contains a portion of a spatial object, otherwise is 0
  - Fig b: when the object is continuous function of time, the pattern vector is obtained at discrete time instance $t_i$, by letting $x_i = f(t_i)$ for $i = 1, ..., n$




Two simple ways of coding patterns into pattern vectors: (a) spatial object and (b) temporal object (waveform).

- Example: for $n = 2$ and $R = 4$
- $X = [20\ 10] \in \S_2$, $X = [4\ 6] \in \S_3$
- The regions denoted by $\S_i$ are called decision regions.
- Regions are seperated by decision surface
  - The patterns on decision surface does not belong to any class
  - decision surface in $E^2$ is curve, for general case, $E^n$ is $(n-1)-$dimentional hepersurface.



# Discriminant Functions

- During the classification, the membership in a category should be determined by classifier based on discriminant functions $g_1(X), ..., g_R(X)$
- Assume $g_i(X)$ is scalar.
- The pattern belongs to the $i$th category iff $g_i(X) > g_j(X)$ for $i, j = 1, ..., R, i \neq j$.
- $\therefore$ within the region $\S_i$, $i$th discriminant function have the largest value.
- Decision surface contain patterns $X$ without membership in any classes
- The decision surface is defined as:

$$g_i(X) - g_j(X) = 0$$

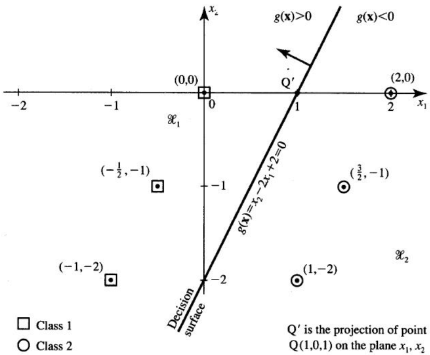
- **Example:** Consider six patterns, in two dimensional pattern space to be classified in two classes:
  $\{[0\ 0]', [-0.5\ \ -1]', [-1\ \ -2]'\}$: class 1
  $\{[2\ 0]', [1.5\ \ -1]', [1\ \ -2]'\}$: class 2
- Inspection of the patterns shows that the $g(X)$ can be arbitrarily chosen

$$g(X) = -2x_1 + x_2 + 2$$

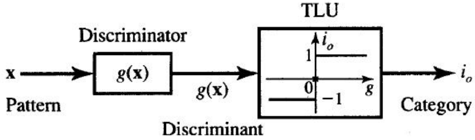
$$g(X) > 0 : \text{class 1}$$

$$g(X) < 0 : \text{class 2}$$

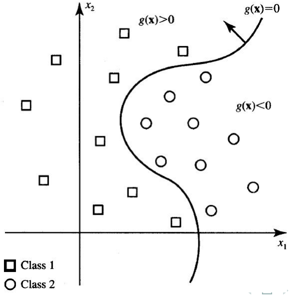
$$g(X) = 0 : \text{on the surface}$$



- The classifier can be implemented as shown in Fig. below (TLU is threshold logic




- **Example 2:** Consider a classification problem as shown in fig. below
- the discriminant surface can not be estimated easily.
- It may result in a nonlinear function of $x_1$ and $x_2$.



# Pattern Classification

- In pattern classification we assume
  - The sets of classes and their members are known
- Having the patterns, We are looking to find the discriminant surface by using NN,
- The only condition is that the patterns are separable
- The patterns like first example are linearly separable and in second example are nonlinearly separable
- In first step, simple sparable systems are considered.

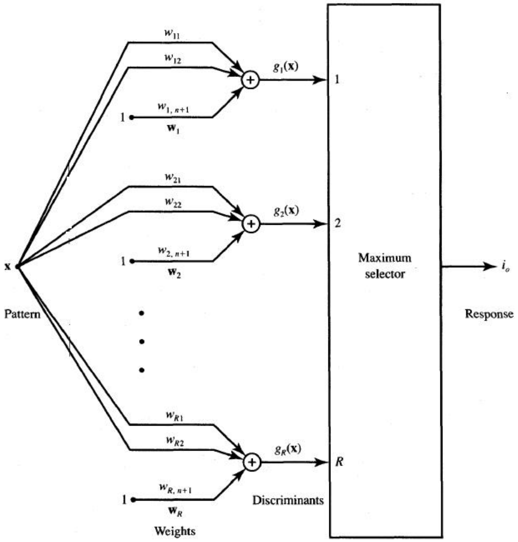
# Linear Machine

- Linear discernment functions are the simplest discriminant functions:
$$g(x) = w_1 x_1 + w_2 x_2 + \ldots + w_n x_n + w_{n+1} \quad (1)$$
- Consider $\mathbf{x} = [x_1, \ldots, x_n]^T$, and $\mathbf{w} = [w_1, \ldots, w_n]^T$, (1) can be redefined as $g(x) = \mathbf{w}^T \mathbf{x} + w_{n+1}$
- Now we are looking for $\mathbf{w}$ and $w_{n+1}$ for classification
- The classifier using the discriminant function (1) is called Linear Machine.
- Minimum distance classifier (MDC) or nearest neighborhood are employed to classify the patterns and find $w$'s:
  - $E^n$ is the $n$-dimensional Euclidean pattern space $\rightsquigarrow$ Euclidean distance between two point are $\|x_i - x_j\| = [(x_i - x_j)^T (x_i - x_j)]^{1/2}$.
  - $P_i$ is center of gravity of cluster $i$.
  - A MDC computes the distance from pattern $x$ of unknown to each prototype ($\|x - p_i\|$).
  - The prototype with smallest distance is assigned to the pattern.

# Linear Machine

- Calculating the squared distance:
  $\|x - p_i\|^2 = (x - p_i)^T (x - p_i) = x^T x - 2x^T p_i + p_i^T p_i$
- $x^T x$ is independent of $i$
- $\therefore$ min the equation above is obtained by max the discernment function: $g_i(x) = x^T p_i - \frac{1}{2} p_i^T p_i$
- We had $g_i(x) = \mathbf{w}_i^T \mathbf{x} + w_{in+1}$
- $\therefore$ Considering $p_i = (p_{i1}, p_{i2}, ..., p_{in})^T$,
- The weights are defined as

$$\begin{aligned} w_{ij} &= p_{ij} \\ w_{in+1} &= -\frac{1}{2} p_i^T p_i, \\ & i = 1, ..., R, j = 1, ..., n \end{aligned} \tag{2}$$




A linear classifier.

# Example

- In this example a linear classifier is designed
- Center of gravity of the prototypes are known a priori

$$p_1 = \begin{bmatrix} 10 \\ 2 \end{bmatrix}, p_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, p_3 = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

- Using (2) for $R = 3$, the weights are

$$w_1 = \begin{bmatrix} 10 \\ 2 \\ -52 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ -5 \\ -14.5 \end{bmatrix}, w_3 = \begin{bmatrix} -5 \\ 5 \\ -25 \end{bmatrix}$$

- Discriminant functions are:

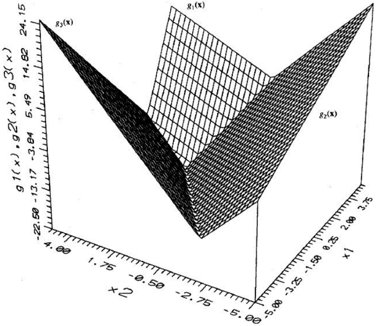
$$\begin{aligned} g_1(x) &= 10x_1 + 2x_2 - 52 \\ g_2(x) &= 2x_1 - 5x_2 - 14.5 \\ g_3(x) &= -5x_1 + 5x_2 - 25 \end{aligned}$$

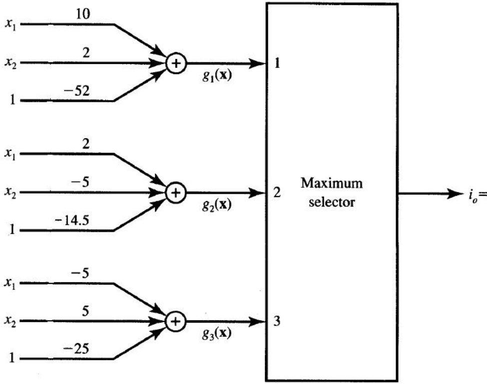
The decision lines are:

$$S_{12} \quad : 8x_1 + 7x_2 - 37.5 = 0$$

$$S_{13} \quad : -15x_1 + 3x_2 + 27 = 0$$

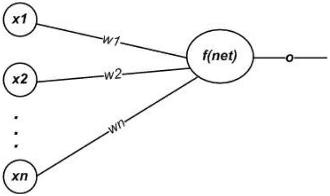
$$S_{23} \quad : -7x_1 + 10x_2 - 10.5 = 0$$



# Bias or Threshold?

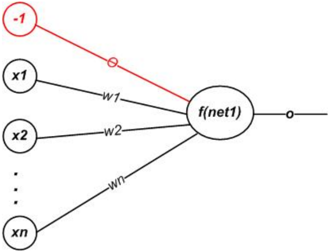
- Revisit the structure of a single layer network




- Considering the threshold ($\theta$) the activation function is defined as

$$o = \begin{cases} 1 & net \geq \theta \\ -1 & net < \theta \end{cases} \quad (3)$$

- Now define $net_1 = net - \theta$:




- ∴ The activation function can be considered as

$$o = \begin{cases} 1 & net_1 \geq 0 \\ -1 & net_1 < 0 \end{cases} \quad (4)$$

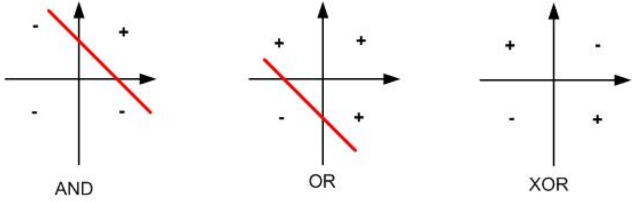
- ∴Bias can be play as a threshold in activation function.
- Considering neither threshold nor bias implies that discriminant function always intersects the origin which is not always correct.

- If $R$ linear functions $g_i(x) = w_1x_1 + w_2x_2 + ... + w_nx_n + w_{n+1}, i = 1, ..., R$ exists s.t

$$g_i(x) > g_j(x) \ \forall x \in \S_i, i, j = 1, ..., R, i \neq j$$

the pattern set is linearly separable

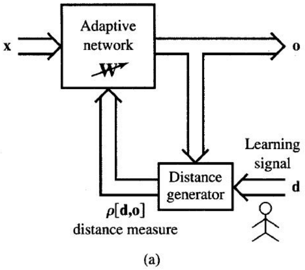
- Single layer networks can only classify linearly separable
- Nonlinearly separable patterns are classified by multiple layer networks
- **Example:** AND: $x_2 = -x_1 + 1 \ \ (b = -1, w_1 = 1, \ w_2 = 1)$
- **Example:** OR: $x_2 = -x_1 - 1 \ \ (b = 1, w_1 = 1, \ w_2 = 1)$



# Learning

- When there is no a priori knowledge of $p_i$'s, a method should be found to adjust weights ($w_i$).
- We should learn the network to behave as we wish
- Learning task is finding $w$ based on the set of training examples $x$ to provide the best possible approximation of $h(x)$.
  - In classification problem $h(x)$ is discriminant function $g(x)$.
- Two types of learning is defined
  1. Supervised learning: At each instant of time when the input is applied the desired response $d$ is provided by teacher
     - The error between actual and desired response is used to correct and adjust the weights.
     - It rewards accurate classifications/ associations and punishes those yields inaccurate response.

2. Unsupervised learning: desired response is not known to improve the network behavior.
   - A proper self-adoption mechanism has to be embedded.




Block diagram for explanation of basic learning modes: (a) supervised learning and (b) unsupervised learning.

- General learning rule is the weight vector $w_i = [w_{i1} \ w_{i2} \ \ldots \ w_{in}]^T$ increases in proportion to the product of input $x$ and learning signal $r$

$$
\begin{aligned}
w_i^{k+1} &= w_i^k + \Delta w_i^k \\
\Delta w_i^k &= c r^k (w_i^k, x^k) x^k
\end{aligned}
$$

$c$ is pos. const.: learning constant.

  - For supervised learning $r = r(w_i, x, d_i)$
  - For continuous learning

$$\frac{dw_i}{dt} = crx$$

# Perceptron Learning Rule

- Perceptron learning rule was first time proposed by Rosenblatt in 1960.
- Learning is supervised.
- The weights are updated based the error between the system output and desired output

$$
\begin{aligned}
r^k &= d^k - o^k \\
\Delta W_i^k &= c(d_i^k - o_i^k)x^k
\end{aligned}
\tag{5}
$$

- Based on this rule weights are adjusted iff output $o_i^k$ is incorrect.
- The learning is repeated until the output error is zero for every training pattern
- It is proven that by using Perceptron rule the network can learn what it can present
  - If there are desired weights to solve the problem, the network weights converge to them.

# Single Discrete Perceptron Training Algorithm

- Given $P$ training pairs $\{x_1, d_1, \; x_2, d_2, \; \dots, x_p, d_p\}$ where $x_i$ is $(n \times 1)$, $d_i$ is $(1 \times 1)$, $i = 1, \dots, P$
- The augmented input vectors are $y_i = \begin{bmatrix} x_i \\ -1 \end{bmatrix}$, for $i = 1, \dots, P$
- In the following, $k$ is training step and $p$ is step counter within training cycle.
  1. Choose $c > 0$
  2. Initialized weights at small random values, $w$ is $(n+1) \times 1$
  3. Initialize counters and error: $k \longleftarrow 1$, $p \longleftarrow 1, E \longleftarrow 0$
  4. Training cycle begins here. Set $y \longleftarrow y_p, d \longleftarrow d_p$, $o = sgn(w^T y)$ (sgn is sign function)
  5. Update weights $w \longleftarrow w + \frac{1}{2}c(d - o)y$
  6. Find error: $E \longleftarrow \frac{1}{2}(d - o)^2 + E$
  7. If $p < P$ then $p \longleftarrow p + 1, k \longleftarrow k + 1$, go to step 4, otherwise, go to step 8.
  8. If $E = 0$ the training is terminated, otherwise $E \longleftarrow 0, p \longleftarrow 1$ go to step 4 for new training cycle

# Convergence of Perceptron Learning Rule

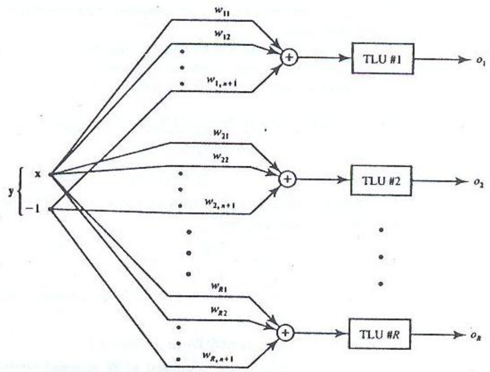
- Learning is finding optimum weights $W^*$ s.t.

$$\begin{cases} {W^*}^T y > 0 & \text{for} x \in \S_1 \\ {W^*}^T y < 0 & \text{for} x \in \S_2 \end{cases}$$

- Training is terminated when there is no error in classification, $(w^* = w^n = w^{n+1})$.
- Assume after $n$ steps learning is terminated.
- It can be shown that $n$ is bounded
- i.e., after limited number of updating, the weights converge to their optimum values

# Multi-category Single Layer Perceptron

- The perceptron learning rule so far was limited for two category classification
- We want to extend it for multigategory classification
- The weight of each neuron (TLU) is updated independent of other weights.
- The k's TLU reponses +1 and other TLU's -1 to indicate class k




*R*-category linear classifier using *R* discrete perceptrons.

# R-Category Discrete Perceptron Training Algorithm

- Given $P$ training pairs $\{x_1, d_1,\ x_2, d_2,\ \dots, x_p, d_p\}$ where $x_i$ is $(n \times 1)$, $d_i$ is $(R \times 1)$, $i = 1, \dots, P$
- The augmented input vectors are $y_i = \begin{bmatrix} x_i \\ -1 \end{bmatrix}$, for $i = 1, \dots, P$
- In the following, $k$ is training step and $p$ is step counter within training cycle.
  1. Choose $c > 0$
  2. Initialized weights at small random values, $W = [w_{ij}]$ is $R \times (n+1)$
  3. Initialize counters and error: $k \longleftarrow 1$, $p \longleftarrow 1, E \longleftarrow 0$
  4. Training cycle begins here. Set $y \longleftarrow y_p, d \longleftarrow d_p$, $o_i = sgn(w_i^T y)$ for $i = 1, \dots, R$ (sgn is sign function)
  5. Update weights $w_i \longleftarrow w_i + \frac{1}{2}c(d_i - o_i)y$ for $i = 1, \dots, R$
  6. Find error: $E \longleftarrow \frac{1}{2}(d_i - o_i)^2 + E$ for $i = 1, \dots, R$
  7. If $p < P$ then $p \longleftarrow p + 1, k \longleftarrow k + 1$, go to step 4, otherwise, go to step 8.
  8. If $E = 0$ the training is terminated, otherwise $E \longleftarrow 0, p \longleftarrow 1$ go to step 4 for new training cycle

## Example

- Revisit the three classes example
- The discriminant values are

| Discriminant | Class 1 $[10\ 2]'$ | Class 2 $[2\ \ -5]'$ | Class 3 $[-5\ 5]'$ |
|---|---|---|---|
| $g_1(x)$ | 52 | -42 | -92 |
| $g_2(x)$ | -4.5 | 14.5 | -49.5 |
| $g_3(x)$ | -65 | -60 | 25 |

- So the thresholds values: $w_{13}$, $w_{23}$, and $w_{33}$ are 52, 14.5, and 25, respectively.
- Assume additional threshold $T_1 = T_2 = T_3 = -2$ so the threshold are changed to 50, 12.5, and 23, respectively.

- Now use perceptron learning rule:
- Consider randomly chosen initial values:
  $w_1^1 = [1 \ -2 \ 0]', \ w_2^1 = [0 \ -1 \ 2]', \ w_3^1 = [1 \ 3 \ -1]'$
- Use the patterns in sequence to update the weights:
  - $y_1$ is input:

$$sgn([1 \ -2 \ 0]\begin{bmatrix} 10 \\ 2 \\ -1 \end{bmatrix}) = 1$$

$$sgn([0 \ -1 \ 2]\begin{bmatrix} 10 \\ 2 \\ -1 \end{bmatrix}) = -1$$

$$sgn([1 \ 3 \ -1]\begin{bmatrix} 10 \\ 2 \\ -1 \end{bmatrix}) = 1^*$$

  - TLU # 3 has incorrect response. So
    $w_1^2 = w_1^1, \ w_2^2 = w_2^1, \ w_3^2 = [1 \ 3 \ -1]' - [10 \ 2 \ -1]' = [-9 \ 1 \ 0]'$

- $y_2$ is input:

$$sgn([1 \ -2 \ 0]\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}) = 1^*$$

$$sgn([0 \ -1 \ 2]\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}) = 1$$

$$sgn([-9 \ 1 \ 0]\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}) = -1$$

  - TLU # 1 has incorrect response. So $w_1^3 = [1\ 2\ 0]' - [2\ -5\ -1]' = [-1\ 3\ 1]',\ w_2^3 = w_2^2,\ w_3^3 = w_3^2$

- $y_3$ is input:

$$sgn([-1\ 3\ 1]\begin{bmatrix}-5\\5\\-1\end{bmatrix}) = 1^*$$

$$sgn([0\ \ -1\ 2]\begin{bmatrix}-5\\5\\-1\end{bmatrix}) = -1$$

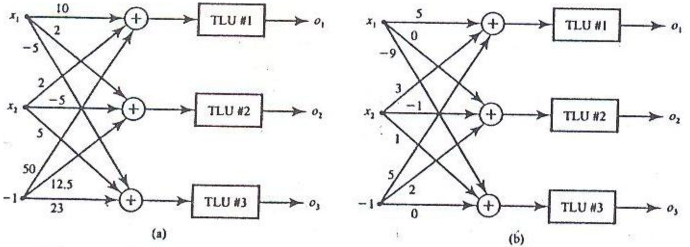
$$sgn([-9\ 1\ 0]\begin{bmatrix}-5\\5\\-1\end{bmatrix}) = 1$$

  - TLU # 1 has incorrect response. So $w_1^4 = [4\ \ -2\ 2]',\ w_2^4 = w_2^3,\ w_3^4 = w_3^3$
- First learning cycle is finished but the error is not zero, so the training is not terminated

- In next training cycles TLU # 2 and 3 are correct.
- TLU # 1 is changed as follows
  $w_1^5 = w_1^4,\ w_1^6 = [2\ 3\ 3]',\ w_1^7 = [7\ \ -2\ 4]',\ w_1^8 = w_1^7,\ w_1^9 = [5\ 3\ 5]$
- The trained network is

$$\begin{aligned} o_1 &= sgn(5x_1 + 3x_2 - 5) \\ o_2 &= sgn(-x_2 - 2) \\ o_1 &= sgn(-9x_1 + x_2) \end{aligned}$$

- The discriminant functions for classification are not unique.




(a) three-perceptron classifier from maxi- mum selector, (b) three-perceptron trained classifier,

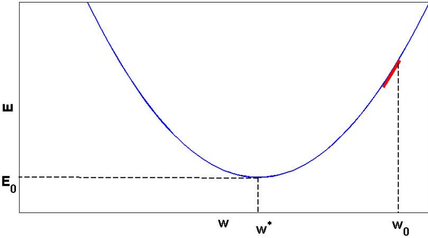
# Continuous Perceptron

- In many cases the output is not necessarily limited to two values ($\pm 1$)
- Therefore, the activation function of NN should be continuous
- The training is indeed defined as adjusting the optimum values of the weights, s.t. minimize a criterion function
- This criterion function can be defined based on error between the network output and desired output.
- Sum of square root error is a popular error function
- The optimum weights are achieved using gradient or steepest decent procedure.

- Consider the error function: $E = E^0 + \lambda(w - w^*)^2 \rightsquigarrow \frac{dE}{dw} = 2\lambda(w - w^*), \frac{d^2E}{dw^2} = 2\lambda$
- The problem is finding $w^*$ s.t min $E$
- To achieve min error at $w = w^*$ from initial weight $w_0$, the weights should move in direction of negative gradient of the curve.
- $\therefore$ The updating rule is

$$w^{k+1} = w^k - \eta \nabla E(w^k)$$

where $\eta$ is pos. const. called learning constant.



- The error to be minimized is

$$E^k = \frac{1}{2}(d^k - o^k)^2$$
$$o^k = f(net^k)$$

- For simplicity superscript $k$ is skipped. But remember the weights updates is doing at $k$th training step.
- The gradient vector is

$$\nabla E(w) = -(d - o)f'(net) \begin{bmatrix} \frac{\partial(net)}{\partial w_1} \\ \frac{\partial(net)}{\partial w_2} \\ \vdots \\ \frac{\partial(net)}{\partial w_{n+1}} \end{bmatrix}$$

- $net = w^T y \rightsquigarrow \frac{\partial(net)}{\partial w_i} = y_i$ for $i = 1, ..., n+1$
- $\therefore \nabla E = -(d - o)f'(net)y$

- The TLU activation function is not useful, since its time derivative is always zero and indefinite at $net = 0$
- Use sigmoid activation function

$$f(net) = \frac{2}{1 + exp(-net)} - 1$$

- Time derivative of sigmoid function can be expressed based on the function itself

$$f'(net) = \frac{2exp(-net)}{(1 + exp(-net))^2} = \frac{1}{2}(1 - f(net)^2)$$

- $o = f(net)$, therefore,

$$\nabla E(w) = -\frac{1}{2}(d - o)(1 - o^2)y$$

- Finally the updating rule is

$$w^{k+1} = w^k + \frac{1}{2}\eta(d^k - o^k)(1 - o^{k2})y^k \tag{6}$$

- Comparing the updating rule of continuous perceprton (6) with the discrete perceptron learning ($w^{k+1} = w^k + \frac{c}{2}(d^k - o^k)y^k$)
  - The correction weights are in the same direction
  - Both involve adding/subtracting a fraction of the pattern vector $y$
  - The essential difference is scaling factor $1 - o^{k2}$ which is always positive and smaller than 1.
  - In continuous learning, a weakly committed perceptron ( *net* close to zero) the correction scaling factor is larger than the more close responses with large magnitude.

# Single Continuous Perceptron Training Algorithm

- Given $P$ training pairs $\{x_1, d_1,\ x_2, d_2,\ ..., x_p, d_p\}$ where $x_i$ is $(n \times 1)$, $d_i$ is $(1 \times 1)$, $i = 1, ..., P$
- The augmented input vectors are $y_i = [x_i \quad -1]^T$, for $i = 1, ..., P$
- In the following, $k$ is training step and $p$ is step counter within training cycle.
  1. Choose $\eta > 0$, $\lambda = 1, E_{max} > 0$
  2. Initialized weights at small random values, $w$ is $(n \times 1) \times 1$
  3. Initialize counters and error: $k \longleftarrow 1$, $p \longleftarrow 1, E \longleftarrow 0$
  4. Training cycle begins here. Set $y \longleftarrow y_p, d \longleftarrow d_p$, $o = f(w^T y)$ (f(net) is sigmoid function)
  5. Update weights $w \longleftarrow w + \frac{1}{2}\eta(d - o)(1 - o^2)y$
  6. Find error: $E \longleftarrow \frac{1}{2}(d - o)^2 + E$
  7. If $p < P$ then $p \longleftarrow p + 1, k \longleftarrow k + 1$, go to step 4, otherwise, go to step 8.
  8. If $E < E_{max}$ the training is terminated, otherwise $E \longleftarrow 0, p \longleftarrow 1$ go to step 4 for new training cycle.

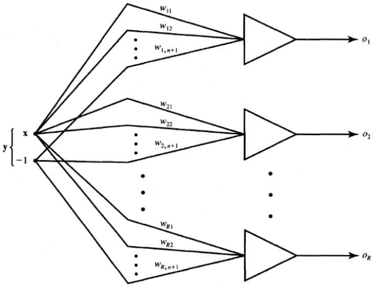
# R-Category Continues Perceptron

- Gradient training rule derived for $R = 2$ is also applicable for multi-category classifier
- The training rule with be changed to

$$w_i^{k+1} = w_i^k + \frac{1}{2}\eta(d_i^k - o_i^k)(1 - o_i^{k2})y^k,$$
$$for \;\; i = 1, ..., R$$

- It is equivalent to individual weight adjustment

$$w_{ij}^{k+1} = w_{ij}^k + \frac{1}{2}\eta(d_i^k - o_i^k)(1 - o_i^{k2})y_j^k,$$
$$for \;\; j = 1, ..., n+1, \;\; i = 1, ..., R$$



R-category linear classifier using continuous perceptrons.