Signals and Systems
Lecture 1: Signals and Systems

Farzaneh Abdollahi

Department of Electrical Engineering

Amirkabir University of Technology

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What are Signals and Systems?

- **Signal:** a function of one or several independent variables which conveys behavioral information of an event
  - Give me some examples :)

What are Signals and Systems?

- **Signal:** a function of one or several independent variables which conveys behavioral information of an event
  - Give me some examples :)
- **System:** It receives signals, processes them and produces novel signals

  - Give me some examples :)

Signal

- In this course we consider signals with one independent variable.
- A signal can be
  - Deterministic: It can be described by a math relation or a table. (Give me an example)
  - Stochastic: It cannot been described by determined math relation. It may be defined by probability density function, distribution function or etc. (Give me an example)
- A signal also can be
  - Continuous Time (x(t)): The independent variable (time) is continuous. (Example!)
  - Discrete Time (x[n]): The independent variable (time) is discrete. (Example!)
    - Digital: Both independent variable and signal domain are discrete.

Signal Energy and Power

- Usually the signals represent physical quantities $\rightsquigarrow$ it captures power and energy
- Consider voltage (v(t)) and current (i(t)) across a resistor (R).
- The instantaneous power is: $p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$
- Total energy over time interval $[t_1, t_2]$: $\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$
- Average power over time interval $[t_1, t_2]$: $\frac{1}{t_2-t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$

Signal Energy and Power

- In general signals may take on complex values.
- General definition of total energy over time interval $[t_1, t_2]/[n_1, n_2]$:

$$\int_{t_1}^{t_2} |x(t)|^2 dt \qquad \sum_{n_1}^{n_2} |x[n]|^2$$

- General definition of average power over time interval $[t_1, t_2]/[n_1, n_2]$:

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt \qquad \frac{1}{n_2 - n_1 + 1} \sum_{n_1}^{n_2} |x[n]|^2$$

Signal Energy and Power

- General definition of total energy over time interval $[-\infty, \infty]/[-\infty, \infty]$:

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt \qquad \lim_{N\to\infty} \sum_{-N}^{N} |x[n]|^2$$

- General definition of average power over time interval $[-\infty, \infty]/[-\infty, \infty]$:

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \qquad \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2$$

Signal Energy and Power

- Based on signal energy and power, a signal may have:
  - Finite energy, zero average power
  - Finite average power, infinite energy
  - Both P_∞ and E_∞ are infinite

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    - Example: $x(t) = 5t^2$

Transformations of Independent Variable (time axis)

- **Time Shift** ($x(t \pm t_0)$/ $x[n \pm n_0]$)
  t_0 and n_0 are positive.
  1. $x(t - t_0)$/ $x[n - n_0]$
     - It is a delayed version of $x(t)$/ $x[n]$
     - Shift the signal to the <u>right</u> by t_0/ n_0
  2. $x(t + t_0)$/ $x[n + n_0]$
     - It is an advanced version of $x(t)$/ $x[n]$
     - Shift the signal to the <u>left</u> by t_0/ n_0

Transformations of Independent Variable (time axis)

- **Time Reversal** ($x(-t)$ / $x[-n]$)
  - Reflects the signal about $t = 0$ / $n = 0$
- **Time Scaling** ($x(at)$ / $x[an]$)
  1. $|a| > 1$
     - It is compressed
  2. $|a| < 1$
     - It is stretched
     - For discrete signal, n/a should be an integer

Transformations of Independent Variable (time axis)

- In general form ($x(\alpha t + \beta)$ / $x[\alpha n + \beta]$)
  1. Shift the signal by β
     - $\beta > 0$: shift to left
     - $\beta < 0$: shift to right
  2. Scale or reverse the signal by α
     - $|\alpha| > 1$: compress
     - $|\alpha| < 1$ stretch
     - $\alpha < 0$ reverse

Signal Properties

- **Periodic Signals**
  - $\exists T > 0 s.t.\ x(t) = x(t + T)$
  - $\exists N > 0$ & $integer\ s.t.\ x[n] = x[n + N]$
  - The fundamental period (T_0/N_0) is the smallest positive value of T/N which the above equality holds
  - Is a constant signal periodic?
  - If a signal is not periodic it is called *aperiodic*

Signal Properties

- **Even and Odd signals**
  - **Even Signal:** It is identical to its time-reversed (its reflection about the origin)
    $x(-t) = x(t)$
    $x[-n] = x[n]$
  - **Odd Signal:** It is symmetric about the origin
    $x(-t) = -x(t)$
    $x[-n] = -x[n]$
    It should be 0 at (t=0/n=0)
  - Any signal can be broken to sum of an odd and an even signal:
    - Even part: $E\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$
    - Odd part: $O\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$

Continuous-Time Complex Exponential Signals $x(t) = Ce^{at}$

- Based on C and a the signal has different behavior
  1. C and a are real
     - $a > 0 \rightsquigarrow$ growing signal
     - $a < 0 \rightsquigarrow$ decaying signal
     - $a = 0 \rightsquigarrow$ constant signal

2. C is real and a is purely imaginary ($x(t) = e^{jw_0 t}$)
   - It is periodic:

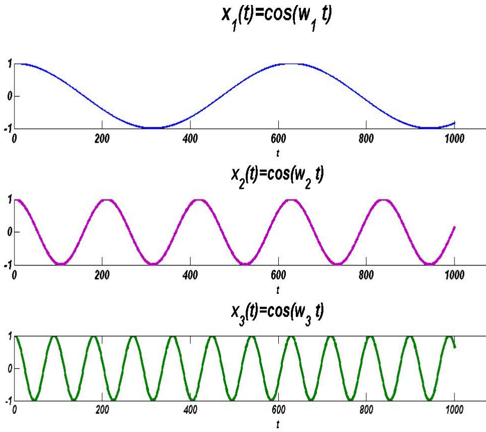
$$e^{jw_0 t} = e^{jw_0 (t+T)} = e^{jw_0 t} e^{jw_0 T} \rightsquigarrow e^{jw_0 T} = 1$$

   - If $w_0 = 0 \rightsquigarrow$ It is periodic for any value of T
   - If $w_0 \neq 0 \rightsquigarrow w_0 T = 2\pi \rightsquigarrow T_0 = \frac{2\pi}{|w_0|}$
   - $e^{jw_0 t}$ and $e^{-jw_0 t}$ have the same fundamental period.

- Cosinusoidal signal $x(t) = ACos(w_0 t + \phi)$
  - Sinusodial signals and exponential signals can be used for expressing physical systems which conserve energy. Example?
  - It is periodic: $T_0 = \frac{2\pi}{w_0}$, (T_0: Fundamental period; w_0 : Fundamental frequency
  - Complex exponential signal can be written in terms of sinusoidal signals with same fundamental period:(Use Euler's relation) $e^{jw_0 t} = Cosw_0 t + jsinw_0 t$
  - Sinusoidal signals can be expressed by complex exponential signals with same fundamental period: $ACos(w_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{jw_0 t} + \frac{A}{2}e^{-j\phi}e^{-jw_0 t} = A\mathfrak{Re}\{e^{j(w_0 t+\phi)}\}$ $ASin(w_0 t + \phi) = A\mathfrak{Im}\{e^{j(w_0 t+\phi)}\}$

Cosinusoidal Signals/ Exponential Signals

- In continuous-time (CT) Cosinusoidal(sinusoidal) signals: $w_0 \uparrow \Rightarrow T_0 \downarrow$
  $w_1 < w_2 < w_3$



Cosinusoidal Signals/ Exponential Signals

- They have finite average power and infinite total energy:
  - In one period: $E_{period} = \int_0^{T_0} |e^{jw_0 t}|^2 dt = T_0, P_{period} = \frac{1}{T_0} E_{period} = 1$
  - By repeating T_0 infinite time: E_∞ becomes infinite, $P_\infty = 1$
- They can be applied in making a set of harmonically related complex exponentials with common period T_0
  - $\phi_k(t) = e^{jkw_0 t}, \ k = 0, \pm 1, \pm 2, \ldots$
  - e^{jwt} is periodic with period T_0:
    $e^{jwT_0} = 1 \rightsquigarrow wT_0 = 2\pi k, w_0 = \frac{2\pi}{T_0}, \ k = 0, \pm 1, \pm 2, \ldots$
  - The kth harmonic is periodic with fundamental freq. $|k|w_0$

Time Shift ⇔ Change Phase

- Time Shift: $ACos(\omega_0(t + t_0) + \phi) = ACos(\omega_0 t + \underbrace{\omega_0 t_0}_{\phi_0} + \phi)$ Change phase

Time Shift $\Leftrightarrow$ Change Phase

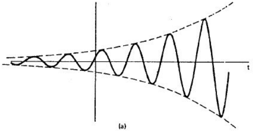
- Time Shift: $ACos(\omega_0(t + t_0) + \phi) = ACos(\omega_0 t + \underbrace{\omega_0 t_0}_{\phi_0} + \phi)$ Change phase
- Change phase: $ACos(\omega_0 t + \phi + \phi_0) = ACos(\omega_0 t + \omega_0 t_0 + \phi)$ Time Shift

Time Shift ⇔ Change Phase

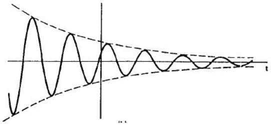
- Time Shift: $ACos(\omega_0(t+t_0)+\phi) = ACos(\omega_0 t + \underbrace{\omega_0 t_0}_{\phi_0} + \phi)$ Change phase
- Change phase: $ACos(\omega_0 t + \phi + \phi_0) = ACos(\omega_0 t + \omega_0 t_0 + \phi)$ Time Shift
- $Cos(\omega_0 t)$ is an Even signal $\longrightarrow$ $Cos(\omega_0 t - \frac{\pi}{2}) = Sin(\omega_0)$ is an Odd Signal

3. Both C and a are complex

- where $C = |C|e^{j\theta}$, $a = r + jw_0 \rightsquigarrow Ce^{at} = |C|e^{rt}e^{j(w_0 t+\theta)} = |C|e^{rt}(Cos(w_0 t + \theta) + jSin(w_0 t + \theta))$
  - $r = 0$: both real and imaginary terms are sinusoidal
  - $r > 0$:real and imaginary terms are increasing

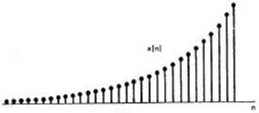


  - $r < 0$: real and imaginary terms are decreasing



Discontinuous-Time (DT) Complex Exponential Signals $x[n] = Ca^n = Ce^{\beta n}$

1. *a* and *C* are real
   - $a > 1$:

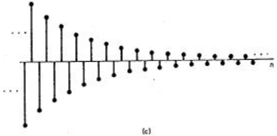


(a)

   - $0 < a < 1$:

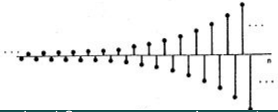
(b)

   - $-1 < a < 0$:



(c)

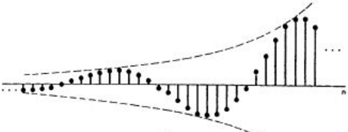
   - $a < -1$:



2. C is real and a is purely imaginary ($x[n] = e^{jw_0 n}$)
   - Similar to CT exponential signals it is related to Cosinusoidal signals: $x[n] = Acos(w_0 n + \phi)$
     - By using Euler's relation: $e^{jw_0 n} = Cosw_0 n + jSinw_0 n$
       $ACos(w_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{jw_0 n} + \frac{A}{2} e^{-j\phi} e^{-jw_0 n}$
   - Similar to CT exponential signals its average power is finite (1) and its total energy is infinite

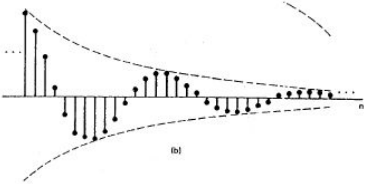
3. Both C and a are complex

- where $C = |C|e^{j\theta}$, $a = |a|e^{jw_0} \rightsquigarrow Ca^n = |C||a|^n e^{j(w_0 n+\theta)} = |C||a|^n(Cos(w_0 n + \theta) + jSin(w_0 n + \theta))$
  - $|a| = 1$: both real and imaginary terms are sinusoidal
  - $|a| > 1$:real and imaginary terms are increasing



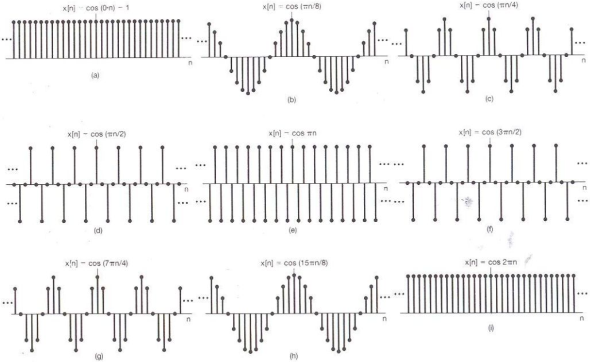
(a)

  - $|a| < 1$: real and imaginary terms are decreasing



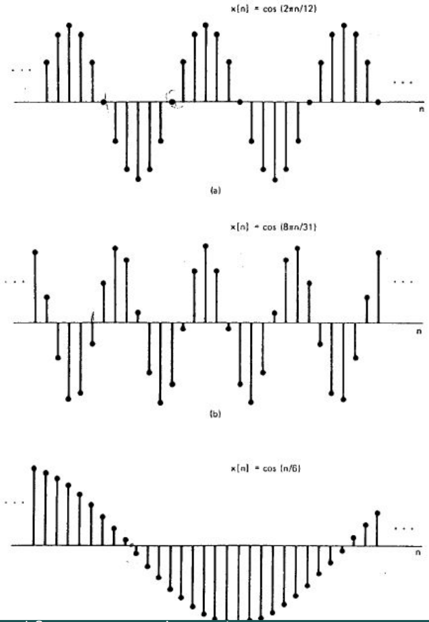
(b)

- $w_0 \uparrow$:
  - $0 < w_0 < \pi \rightsquigarrow$ rate of oscillation $\uparrow$
  - $\pi < w_0 < 2\pi \rightsquigarrow$ rate of osc. $\downarrow$
- $w_0 = \pi$ has highest rate of osc. $(e^{j\pi n} = (-1)^n)$
- Rate of osc. for $w_0 = 0$ equals to $w_0 = 2\pi$



- To have periodic DT exponential signal: $e^{jw_0 n} = e^{jw_0(n+N)}$
- $e^{jw_0 N} = 1 \rightsquigarrow Nw_0 = 2\pi m$
- m, N should be an integer
- Fundamental period $N = \frac{2\pi m}{w_0}$

- $x[n] = Cos(\frac{2\pi n}{12})$ and $x(t) = Cos(\frac{2\pi t}{12})$ are periodic ($T_0 = N = 12$)
- $x(t) = Cos(\frac{8\pi t}{31})$ is periodic ($T_0 = \frac{31}{4}$)
- $x[n] = Cos(\frac{8\pi n}{31})$ is periodic($N = 31$)
- $x(t) = Cos(\frac{t}{6})$ is periodic ($T_0 = \frac{12}{\pi}$)
- But $x[n] = Cos(\frac{n}{6})$ is not periodic!
- $N = 12\pi m$: no integer m can be found to make N an integer



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- Change phase: $ACos(\omega_0 n + \phi + \phi_0) \stackrel{?}{=} ACos(\omega_0 n + \omega_0 n_0 + \phi)$ Not Always Time Shift

Compare $e^{j\omega_0 t}$ in CT and DT

$e^{j\omega_0 t}$ in CT	$e^{j\omega_0 n}$ in DT
Distinct Signal for Distinct value of ω_0 $\omega_1 \neq \omega_2 \rightsquigarrow ACos(\omega_1 t) \neq ACos(\omega_2 t)$	Identical Signals for exponentials at frequencies separated by 2π $\omega_1 \neq \omega_2$ If $\omega_2 = \omega_1 + 2\pi m$ $\rightsquigarrow ACos(\omega_1 n) = ACos(\omega_2 n)$
Periodic for any choice of ω_0	Only periodic if $\omega_0 = \frac{2\pi m}{N}$ for some integers $N > 0, m$
Fundamental frequency ω_0	Fundamental frequency $\frac{\omega_0}{m}$
Fundamental Period $\omega_0 = 0$ undefined $\omega_0 \neq 0$ $\frac{2\pi}{\omega_0}$	Fundamental Period $\omega_0 = 0$ undefined $\omega_0 \neq 0$ $m(\frac{2\pi}{\omega_0})$

- Harmonically related complex exponentials with common period N
- DT exponential signals can make harmonically related sets : $\phi_k[n] = e^{jk(\frac{2\pi}{N})n}$
- BUT we have only N different harmonic $(k = 0, \ldots, \pm(N-1))$
  - $e^{j(k+N)(\frac{2\pi}{N})n} = e^{jk(\frac{2\pi}{N})n} e^{j2\pi n} = e^{jk(\frac{2\pi}{N})n}$
  - $\therefore \phi_N[n] = \phi_0[n], \ldots \phi_{-1}[n] = \phi_{N-1}[n], \ldots$

Unit Step and Unit Impulse

- DT unit impulse: $\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$
- DT unit step: $u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$
- Unit impulse can be defined based on unite step: $\delta[n] = u[n] - u[n-1]$
- Unit step can be described based on unite impulse: $u[n] = \sum_{m=-\infty}^{n} \delta[m]$
  - One can also define it as: $u[n] = \sum_{k=\infty}^{0} \delta[n-k] = \sum_{k=0}^{\infty} \delta[n-k]$
- Unite impulse has sampling property: $x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$

Unit Step and Unit Impulse

- CT unit step:
  $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$
  - It is discontinuous (at t=0)
- Impulse Signal
  $\delta(t) = \frac{du(t)}{dt}$
- $\delta(t) = \lim_{\Delta \to 0} \delta_{\Delta}(t)$

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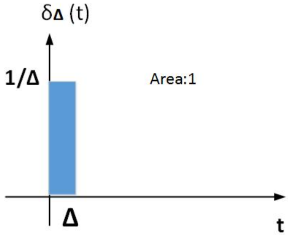
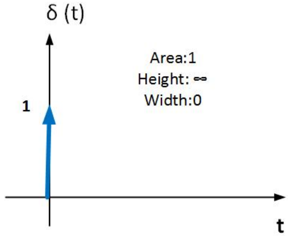




Unit Step and Unit Impulse

- Unit step can be described based on unite impulse: $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$
  - One can also define it as: $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = \int_{0}^{\infty} \delta(t-\sigma) d\sigma$
- Unite impulse has sampling property: $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$

Systems

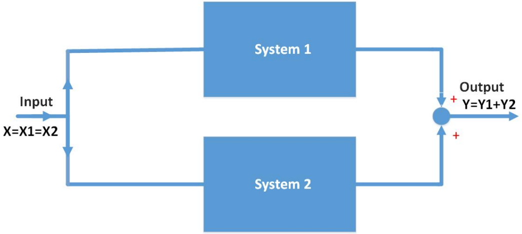
- A system provides output signals by processing input signals
  - Continuous-Time systems: Both input and output signals are continuous.

  - Discrete-Time systems: Both input and output signals are discrete.

Interconnection Systems

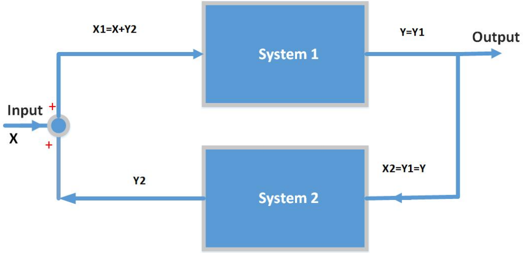
- Some systems consist of some interconnected subsystems:
  - Series (Cascade) interconnection: Output of first subsystem is input of the second subsystem; output of system is output of subsystem 2.

  - Parallel interconnection: Both subsystems receive the same input signal and Output the system is sum of subsystem 1 and subsystem 2.



Interconnection Systems

- Feedback interconnection: Output of subsystem 1 is be input of subsystem 2; output of subsystem 2 is fed back and added to external input to make subsystem 1 input.



System Properties

- System with/without memory:
  - Output of a memoryless system for each value of the independent variable at a given time depends on the input at the same time. (Example?)
  - If output of a system at a given time depends on either future or past time input, it is called system with memory
    - In physical systems memory of a system is related to storing energy like a capacitor in electrical circuit.

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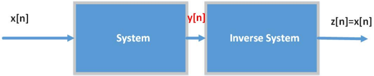
    - In physical systems memory of a system is related to storing energy like a capacitor in electrical circuit.
- Invertibility
  - In invertible system distinct inputs lead to distinct output.
  - If an inverse exists, by cascading it with the original system, the resulting output equals to input of the first system



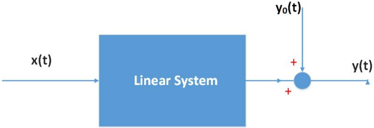
- Does cascading two invertible subsystem yield to invertible system?

System Properties

- Causality
  - Output of a causal system depends on the present and past time.
  - They are called nonpredictive systems.
  - All memoryless systems are causal
- Stability
  - Different type of stability is defined for a system such as I/s stability, asymptotic stability and...
  - In this course Bounded Input Bounded Output (BIBO) stability is considered:
    - small input does not make the output diverge.
    - A counterexample is enough to show a system is unstable.
    - To show the stability one should prove it through the stability definition.
- Time invariancy
  - The characteristic of a time invariant system does not change over time:
    - If $x(t) \rightarrow y(t) \Rightarrow x(t-t_0) \rightarrow y(t-t_0)$
    - If $x[n] \rightarrow y[n] \Rightarrow x[n-n_0] \rightarrow y[n-n_0]$
  - If time is explicitly mentioned in system model it is called time varying

System Properties

- Linearity
  - For linear system superposition property is hold:
    - Additivity property: $x_1 + x_2 \Rightarrow y_1 + y_2$
    - scaling(homogeneity) property: for a complex a, $ax \Rightarrow ay$
  - For linear systems $x(t) = 0(x[n] = 0) \rightsquigarrow y(t) = 0(y[n] = 0)$
- Incrementally linear system consists of a linear system adding with a zero-input response.



  - The difference between the responses to any two inputs will be a linear function of the difference between two inputs
  - Example: $y(t) = 5x(t) + 6$