Nonlinear Control
Lecture 2:Phase Plane Analysis

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- **Phase Plane Analysis**: is a graphical method for studying second-order systems by
  - providing motion trajectories corresponding to various initial conditions.
  - then examine the qualitative features of the trajectories.
  - finally obtaining information regarding the stability and other motion patterns of the system.
- It was introduced by mathematicians such as Henri Poincare in 19th century.

http://en.wikipedia.org/wiki/Henri_Poincar%C3%A9

Motivations

- Importance of Knowing **Phase Plane Analysis:**
  - Since it is on second-order, the solution trajectories can be represented by carves in plane $\rightsquigarrow$ provides easy visualization of the system qualitative behavior.
  - Without solving the nonlinear equations analytically, one can study the behavior of the nonlinear system from various initial conditions.
  - It is not restricted to small or smooth nonlinearities and applies equally well to strong and hard nonlinearities.
  - There are lots of practical systems which can be approximated by second-order systems, and apply phase plane analysis.
- Disadvantage of **Phase Plane Method:** It is restricted to at most second-order and graphical study of higher-order is computationally and geometrically complex.

Concept of Phase Plane

- Phase plane method is applied to autonomous 2*nd* order system described as follows:

$$\dot{x}_1 = f_1(x_1, x_2) \quad (1)$$
$$\dot{x}_2 = f_2(x_1, x_2) \quad (2)$$

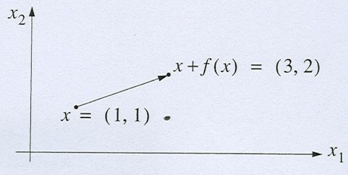
- $f_1, \ f_2 : \mathcal{R}^2 \rightarrow \mathcal{R}$.
- System response $(x(t) = (x_1(t), \ x_2(t)))$ to initial condition $x_0 = (x_{10}, \ x_{20})$ is a mapping from $\mathcal{R}$ to $\mathcal{R}^2$.
- The $x_1 - x_2$ plane is called **State plane** or **Phase plane**
- The locus in the $x_1 - x_2$ plane of the solution $x(t)$ for all $t \geq 0$ is a curve named **trajectory** or **orbit** that passes through the point x_0
- The family of phase plane trajectories corresponding to various initial conditions is called **Phase protrait** of the system.

How to Construct Phase Plane Trajectories?

- Despite of exiting several routines to generate the phase portraits by computer, it is useful to learn roughly sketch the portraits or quickly verify the computer outputs.
- Some methods named: Isocline, Vector field diagram, delta method, Pell's method, etc
- Vector Field Diagram:
  - Revisiting (1) and (2):

$$\dot{x} = f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}, \dot{x} = (\dot{x}_1, \ \dot{x}_2)$$

  - To each vector (x_1, x_2), a corresponding vector $(f_1(x_1, x_2), \ f_2(x_1, x_2))$ known as a **vector field** is associated.
  - Example: If $f(x) = (2x_1^2, x_2)$, for $x = (1, 1)$, next point is $(1, 1) + (2, 1) = (3, 2)$

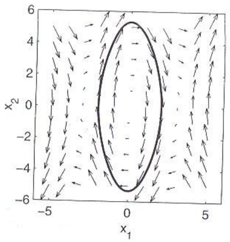


Vector field representation.

Vector Field Diagram

- By repeating this for sufficient point in the state space, a **vector field diagram** is obtained.
- Noting that $\frac{dx_2}{dx_1} = \frac{f_2}{f_1} \rightsquigarrow$ vector field at a point is tangent to trajectory through that point.
  - $\therefore$ starting from x_0 and by using the vector field with sufficient points, the trajectory can be constructed.
- Example: Pendulum without friction

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10 \sin x_1 \end{aligned}$$



Vector field diagram of the pendulum equation without friction.

Isocline Method

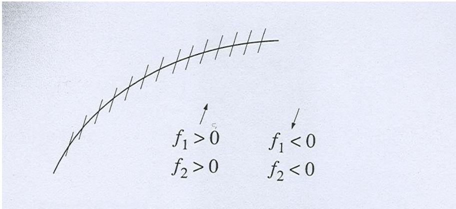
- The term isocline derives from the Greek words for "*same slope*."
- Consider again Eqs (1) and (2), the slope of the trajectory at point x:

$$S(x) = \frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

- An isocline with slope α is defined as $S(x) = \alpha$
- $\therefore$ all the points on the curve $f_2(x_1, x_2) = \alpha f_1(x_1, x_2)$ have the same tangent slope α.
- Note that the "time" is eliminated here $\Rrightarrow$ The responses $x_1(t)$ and $x_2(t)$ cannot be obtained directly.
- Only qualitative behavior can be concluded, such as stable or oscillatory response.

Isocline Method

- The algorithm of constructing the phase portrait by isocline method:
  1. Plot the curve $S(x) = \alpha$ in state-space (phase plane)
  2. Draw small line with slope α. Note that the direction of the line depends on the sign of f_1 and f_2 at that point.

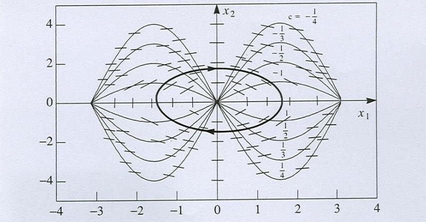


Isocline with positive slope.

  3. Repeat the process for sufficient number of α s.t. the phase plane is full of isoclines.

Example: Pendulum without Friction

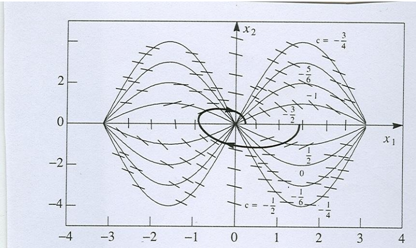
- Consider the dynamics $\dot{x}_1 = x_2, \quad \dot{x}_2 = -sinx_1 \quad \therefore S(x) = \frac{-sinx_1}{x_2} = c$
- Isoclines: $x_2 = \frac{-1}{c} sinx_1$
- Trajectories for different init. conditions can be obtained by using the given algorithm
- The response for $x_0 = (\frac{\pi}{2}, \ 0)$ is depicted in Fig.
- The closed curve trajectory confirms marginal stability of the system.



Graphical construction of the phase portrait of the pendulum equation (without friction) by the isocline method.

Example: Pendulum with Friction

- Dynamics of pendulum with friction:
  $\dot{x}_1 = x_2, \quad \dot{x}_2 = -0.5x_2 - sin x_1 \quad \therefore S(x) = \frac{-0.5 - sin x_1}{x_2} = c$
- Isoclines: $\quad x_2 = \frac{-1}{0.5+c} sin x_1$
- Similar Isoclines but with different slopes
- Trajectory is drawn for $x_0 = (\frac{\pi}{2}, \ 0)$
- The trajectory shrinks like an spiral converging to the origin



Graphical construction of the phase portrait of the pend
equation (with friction) by the isocline method.

Qualitative Behavior of Linear Systems

- First we analyze the phase plane of linear systems since the behavior of nonlinear systems around equilibrium points is similar of linear ones
- For LTI system:
  $\dot{x} = Ax, \quad A \in \mathcal{R}^{2\times 2}, \; x_0$: initial state $\rightsquigarrow x(t) = Me^{J_r t}M^{-1}x_0$
  J_r : Jordan block of A, M : Matrix of eigenvectors $M^{-1}AM = J_r$
- Depending on the eigenvalues of A, J_r has one of the following forms:

$$\lambda_i: \text{ real \& distinct} \rightsquigarrow J_r = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\lambda_i: \text{ real \& multiple} \rightsquigarrow J_r = \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}, \; k = 0, 1,$$

$$\lambda_i: \text{ complex} \rightsquigarrow J_r = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

- The system behavior is different at each case

Case 1: $\lambda_1 \neq \lambda_2 \neq 0$

- In this case $M = [v_1 \;\; v_2]$ where v_1 and v_2 are real eigenvectors associated with λ_1 and λ_2
- To transform the system into two decoupled first-order diff equations, let $z = M^{-1}x$:

$$\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 \\ \dot{z}_2 &= \lambda_2 z_2 \end{aligned}$$

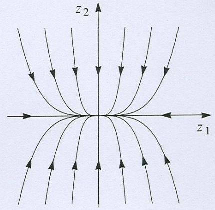
- The solution for initial states $(z_{01}, \; z_{02})$:

$$\begin{aligned} z_1(t) &= z_{10} e^{\lambda_1 t}, \quad z_2(t) = z_{20} e^{\lambda_2 t} \\ \text{eliminating} \quad t \rightsquigarrow z_2 &= C z_1^{\lambda_2/\lambda_1}, \quad C = z_{20}/(z_{10})^{\lambda_2/\lambda_1} \end{aligned} \qquad (3)$$

- Phase portrait is obtained by changing $C \in \mathcal{R}$ and plotting (3).
- The phase portrait depends on the sign of λ_1 and λ_2.

Case 1.1: $\lambda_2 < \lambda_1 < 0$

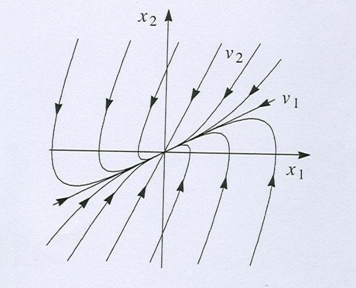
- $t \to \infty$ $\Rightarrow$ the terms $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ tend to zero
  - Trajectories from entire state-space tend to origin $\rightsquigarrow$ the equilibrium point $x = 0$ is **stable node**.
- $e^{\lambda_2 t} \to 0$ faster $\rightsquigarrow$ λ_2 is fast eignevalue and v_2 is fast eigenvector.
- Slope of the curves: $\frac{dz_2}{dz_1} = C \frac{\lambda_2}{\lambda_1} z_1^{(\lambda_2/\lambda_1 - 1)}$
- $\lambda_2 < \lambda_1 < 0 \rightsquigarrow \lambda_2/\lambda_1 > 1$, so slope is
  - **zero** as $z_1 \longrightarrow 0$
  - **infinity** as $z_1 \longrightarrow \infty$.
- $\therefore$ The trajectories are
  - tangent to z_1 axis, as they approach to origin
  - parallel to z_2 axis, as they are far from origin.



Phase portrait of a stable node in modal coo

Case 1.1: $\lambda_2 < \lambda_1 < 0$

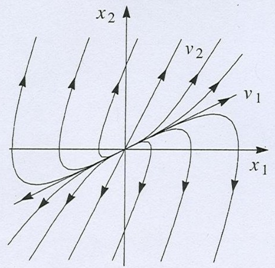
- Since z_2 approaches to zero faster than z_1, trajectories are sliding along z_1 axis
- In X plane also trajectories are:
  - tangent to the slow eigenvector v_1 for near origin
  - parallel to the fast eigenvector v_2 for far from origin



Phase portrait of a stable node.

Case 1.2: $\lambda_2 > \lambda_1 > 0$

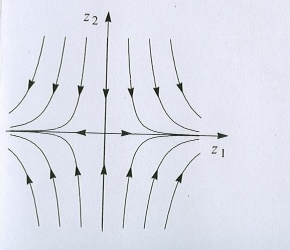
- $t \rightarrow \infty \Rightarrow$ the terms $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ grow exponentially, so
  - The shape of the trajectories are the same, with opposite directions
  - The equilibrium point is socalled unstable node



Phase portrait of an unstable node.

Case 1.3: $\lambda_2 < 0 < \lambda_1$

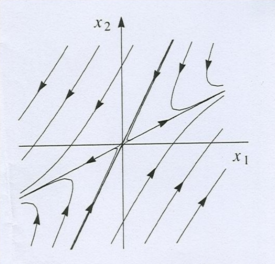
- $t \to \infty \Rightarrow e^{\lambda_2 t} \longrightarrow 0$, but $e^{\lambda_1 t} \longrightarrow \infty$,so
  - λ_2 : stable eigenvalue, v_2: stable eigenvector
  - λ_1 : unstable eigenvalue, v_1: unstable eigenvector
- Trajectories are negative exponentials since $\frac{\lambda_2}{\lambda_1}$ is negative.
- Trajectories are
  - decreasing in z_2 direction, but increasing in z_1 direction
  - tangent to z_1 as $|z_1| \to \infty$ and tangent to z_2 as $|z_1| \to 0$



Phase portrait of a saddle point in modal coordina

Case 1.3: $\lambda_2 < 0 < \lambda_1$

- The exceptions of this hyperbolic shape:
  - two trajectories along z_2-axis $\rightarrow 0$ as $t \rightarrow 0$, called stable trajectories
  - two trajectories along z_1-axis $\rightarrow \infty$ as $t \rightarrow 0$, called unstable trajectories
- This equilibrium point is called saddle point
- Similarly in X plane, stable trajectories are along v_2, but unstable trajectories are along the v_1
- For $\lambda_1 < 0 < \lambda_2$ the direction of the trajectories are changed.



Phase portrait of a saddle point.

Case 2: Complex Eigenvalues, $\lambda_{1,2} = \alpha \pm j\beta$

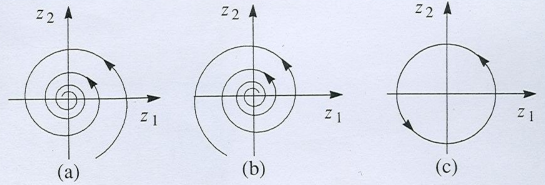
$$\dot{z}_1 = \alpha z_1 - \beta z_2$$
$$\dot{z}_2 = \beta z_1 + \alpha z_2$$

- The solution is oscillatory $\Longrightarrow$ polar coordinates $(r = \sqrt{z_1^2 + z_2^2}, \quad \theta = \tan^{-1}(\frac{z_2}{z_1}))$

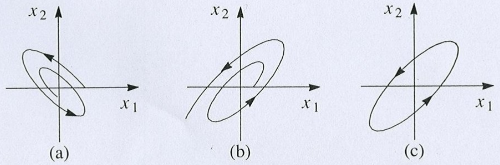
$$\dot{r} = \alpha r \rightsquigarrow r(t) = r_0 e^{\alpha t}$$
$$\dot{\theta} = \beta \rightsquigarrow \theta(t) = \theta_0 + \beta t$$

- This results in Z plane is a logarithmic spiral where α determines the form of the trajectories:
  - $\alpha < 0$: as $t \to \infty \rightsquigarrow r \to 0$ and angle θ is rotating. The spiral converges to origin $\Longrightarrow$ Stable Focus.
  - $\alpha > 0$: as $t \to \infty \rightsquigarrow r \to \infty$ and angle θ is rotating. The spiral diverges away from origin $\Longrightarrow$ Unstable Focus.
  - $\alpha = 0$: Trajectories are circles with radius $r_0 \Longrightarrow$ Center

Case 2: Complex Eigenvalues, $\lambda_{1,2} = \alpha \pm j\beta$



Typical trajectories in the case of complex eigenvalues. (a) $\alpha < 0$; (b) $\alpha > 0$; (c) $\alpha = 0$.



Phase portraits for (a) a stable focus; (b) an unstable focus; (c) a center.

Case 3: Nonzero Multiple Eigenvalues $\lambda_1 = \lambda_2 = \lambda \neq 0$

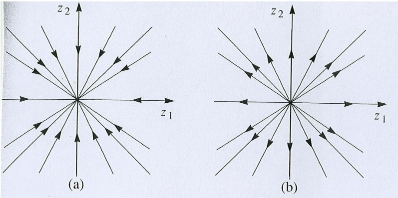
- Let $z = M^{-1}x$:

$$\dot{z}_1 = \lambda z_1 + k z_2, \quad \dot{z}_2 = \lambda z_2$$

the solution is :$z_1(t) = e^{\lambda t}(z_{10} + k z_{20} t), \quad z_2(t) = z_{20} e^{\lambda t} \rightsquigarrow$

$$z_1 = z_2 \left[\frac{z_{10}}{z_{20}} + \frac{k}{\lambda} ln \left(\frac{z_2}{z_{20}} \right) \right]$$

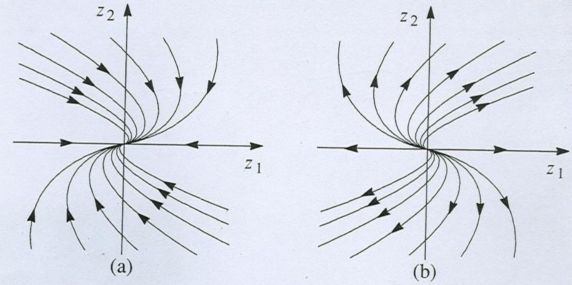
- Phase portrait are depicted for $k = 0$ and $k = 1$.
- When the eignevectors are different $\rightsquigarrow k = 0$:
  - similar to Case 1, for $\lambda < 0$ is stable, $\lambda > 0$ is unstable.
  - Decaying rate is the same for both modes ($\lambda_1 = \lambda_2$) $\rightsquigarrow$ trajectories are lines



Phase portraits for the case of nonzero multiple eigenvalues when $k = 0$: (a) $\lambda < 0$; (b) $\lambda > 0$.

Case 3: Nonzero Multiple Eigenvalues $\lambda_1 = \lambda_2 = \lambda \neq 0$

- There is no fast-slow asymptote.
- $k = 1$ is more complex, but it is still similar to Case 1:



Phase portraits for the case of nonzero multiple eigenvalues when $k = 1$: (a) $\lambda < 0$; (b) $\lambda > 0$.

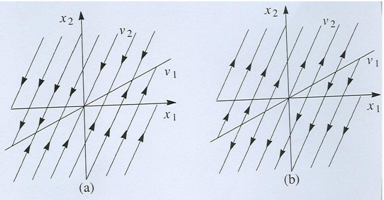
Case 4.1: One eigenvalue is zero $\lambda_1 = 0, \quad \lambda_2 \neq 0$

- A is singular in this case
- Every vector in null space of A is an equilibrium point
- There is a line (subspace) of equilibrium points
- $M = [v_1 \ v_2], \quad v_1, \ v_2$: corresponding eigenvectors, $v_1 \in \mathcal{N}(A)$.

$$\dot{z}_1 = 0, \quad \dot{z}_2 = \lambda_2 z_2$$

$$\text{solution: } z_1(t) = z_{10}, \quad z_2(t) = z_{20} e^{\lambda_2 t}$$

- Phase portrait depends on sign of λ_2:
  - $\lambda_2 < 0$: Trajectories converge to equilibrium line
  - $\lambda_2 > 0$: Trajectories diverge from equilibrium line



Phase portraits for (a) $\lambda_1 = 0, \ \lambda_2 < 0$; (b) $\lambda_1 = 0, \ \lambda_2 > 0$.

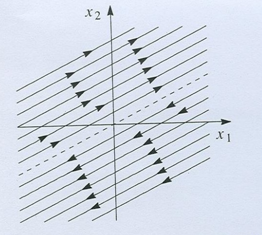
Case 4.2: Both eigenvalues zero $\lambda_1 = \lambda_2 = 0$

- Let $z = M^{-1}x$

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = 0$$

$$\text{solution: } z_1(t) = z_{10} + z_{20}t, \quad z_2(t) = z_{20}$$

- z_1 linearly increases/decreases base on the sign of z_{20}
- z_2 axis is equilibrium subspace in Z-plane
- Dotted line is equilibrium subspace
- The difference between Case 4.1 and 4.2: all trajectories start off the equilibrium set move parallel to it.



Phase portraits when $\lambda_1 = \lambda_2 = 0$.

As Summary:

- Six types of equilibrium points can be identified:
  - stable/unstable node
  - saddle point
  - stable/ unstable focus
  - center
- Type of equilibrium point depends on sign of the eigenvalues
  - If real part of eignevalues are Positive $\rightsquigarrow$ unstability
  - If real part of eignevalues are Negative $\rightsquigarrow$ stability
- All properties for linear systems hold globally
- Properties for nonlinear systems only hold locally

Local Behavior of Nonlinear Systems

- Qualitative behavior of nonlinear systems is obtained locally by linearization around the equilibrium points
- Type of the perturbations and reaction of the system to them determines the degree of validity of this analysis
- A simple example: Consider the linear perturbation case $A \longrightarrow A + \Delta A$, where $\Delta A \in \mathcal{R}^{2\times 2}$: small perturbation
- Eigenvalues of a matrix continuously depend on its parameters
  - Positive (Negative) eigenvalues of A remain positive (negative) under small perturbations.
  - For eigenvalues on the $j\omega$ axis no matter how small perturbation is, it changes the sign of eigenvalue.
- Therefore
  - node or saddle point or focus equilibrium point remains the same under small perturbations
  - This analysis is not valid for a center equilibrium point

- Multiple Equilibria
  - Linear systems can have
    - an isolated equilibrium point or
    - a continuum of equilibrium points (When $detA = 0$)
  - Unlike linear systems, nonlinear systems can have multiple isolated equilibria.
- Qualitative behavior of second-order nonlinear system can be investigated by
  - generating phase portrait of system globally by computer programs
  - linearize the system around equilibria and study the system behavior near them without drawing the phase portrait
    - Let (x_{10}, x_{20}) are equilibrium points of

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \quad (4)$$

    - $f_1,\ f_2$ are continuously differentiable about (x_{10}, x_{20})
    - Since we are interested in trajectories near (x_{10}, x_{20}), define $x_1 = y_1 + x_{10}, \quad x_2 = y_2 + x_{20}$
    - y_1, y_2 are small perturbations form equilibrium point.

Qualitative Behavior Near Equilibrium Points

- Expanding (4) into its Taylor series

$$\dot{x}_1 = \dot{x}_{10} + \dot{y}_1 = \underbrace{f_1(x_{10}, x_{20})}_{0} + \left.\frac{\partial f_1}{\partial x_1}\right|_{(x_{10},x_{20})} y_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{(x_{10},x_{20})} y_2 + H.O.T.$$

$$\dot{x}_2 = \dot{x}_{20} + \dot{y}_2 = \underbrace{f_2(x_{10}, x_{20})}_{0} + \left.\frac{\partial f_2}{\partial x_1}\right|_{(x_{10},x_{20})} y_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_{(x_{10},x_{20})} y_2 + H.O.T.$$

- For sufficiently small neighborhood of equilibrium points, H.O.T. are negligible

$$\begin{cases} \dot{y}_1 = a_{11} y_1 + a_{12} y_2 \\ \dot{y}_2 = a_{21} y_1 + a_{22} y_2 \end{cases}, \quad a_{ij} = \left.\frac{\partial f_i}{\partial x}\right|_{x_0}, \quad i = 1, 2$$

- The equilibrium point of the linear system is $(y_1 = y_2 = 0)$

$$\dot{y} = Ay, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \left.\frac{\partial f}{\partial x}\right|_{x_0}$$

Qualitative Behavior Near Equilibrium Points

- Matrix $\frac{\partial f}{\partial x}$ is called Jacobian Matrix.
- The trajectories of the nonlinear system in a small neighborhood of an equilibrium point are close to the trajectories of its linearization about that point:
- if the origin of the linearized state equation is a
  - stable (unstable) node, or a stable (unstable) focus or a saddle point,
- then in a small neighborhood of the equilibrium point, the trajectory of the nonlinear system will behave like a
  - stable (unstable) node, or a stable (unstable) focus or a saddle point.

Example: Tunnel Diode Circuit

$$\dot{x}_1 = \frac{1}{C}[-h(x_1) + x_2]$$
$$\dot{x}_2 = \frac{1}{L}[-x_1 - Rx_2 + u]$$

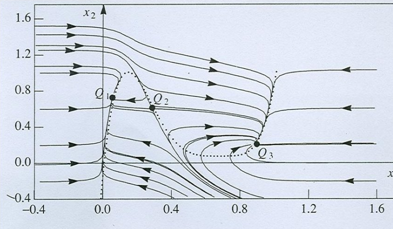
- $u = 1.2v, \; R = 1.5K\Omega, \; C = 2pF, \; L = 5\mu H$, time in nanosecond, current in mA

$$\dot{x}_1 = 0.5[-h(x_1) + x_2]$$
$$\dot{x}_2 = 0.2[-x_1 - 1.5x_2 + 1.2]$$

- Suppose $h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$
- equilibrium points ($\dot{x}_1 = \dot{x}_2 = 0$): $Q_1 = (0.063, 0.758), \; Q_2 = (0.285, 0.61), \; Q_3 = (0.884, 0.21)$

Example: Tunnel Diode Circuit

- The global phase portrait is generated by a computer program is shown in Fig.
- Except for two special trajectories which approach Q_2, all trajectories approach either Q_1 or Q_3.
- Near equilibrium points Q_1 and Q_3 are stable nodes, Q_2 is like saddle point.
- The two special trajectories from a curve that divides the plane into two halves with different behavior (separatrix curves).
- All trajectories originating from left side of the curve approach to Q_1
- All trajectories originating from left side of the curve approach to Q_3



Phase portrait of the tunnel diode circuit

Tunnel Diode:Qualitative Behavior Near Equilibrium Points

- Jacobian matrix

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0.5\acute{h}(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}$$

$$\acute{h}(x_1) = \frac{dh}{dx_1} = 17.76 - 207.58x_1 + 688.86x_1^2 - 905.24x_1^3 + 418.6x_1^4$$

- Evaluate the Jacobian matrix at the equilibriums Q_1, Q_2, Q_3:

$$Q_1 = (0.063, 0.758),\ A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix},\ \lambda_1 = -3.57, \lambda_2 = -0.33 \text{ stable node}$$

$$Q_2 = (0.285, 0.61),\ A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0.2 & -0.3 \end{bmatrix},\ \lambda_1 = 1.77, \lambda_2 = -0.25 \text{ saddle point}$$

$$Q_3 = (0.884, 0.21),\ A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0.2 & -0.3 \end{bmatrix},\ \lambda_1 = -1.33, \lambda_2 = -0.4 \text{ stable node}$$

- $\therefore$ similar results given from global phase portrait.

Tunnel Diode Circuit

- In practice, There are only two stable equilibrium points: Q_1 or Q_3.
- Equilibrium point at Q_2 in never observed,
  - Even if set up the exact initial conditions corresponding t Q_2, the ever-present physical noise causes the trajectory to diverge from Q_2
- Such circuit is called bistable, since it has two steady-state operating points.
- Triggering formQ_1 to Q_3 or vice versa is achieved by changing the load line

- **Special case:** If the Jacobian matrix has eigenvalues on $j\omega$, then the qualitative behavior of nonlinear system near the equilibrium point could be quite distinct from the linearized one.
- Example:

$$\begin{aligned}\dot{x}_1 &= -x_2 - \mu x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= x_1 - \mu x_2(x_1^2 + x_2^2)\end{aligned}$$

- It has equilibrium point at origin $x^* = 0$.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \pm j \Rightarrow \text{ center}$$

- Now consider the nonlinear system

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta \Rightarrow \dot{r} = -\mu r^3, \quad \dot{\theta} = 1$$

- $\therefore$ nonlinear system is stable if $\mu > 0$ and is unstable if $\mu < 0$
- $\therefore$ the qualitative behavior of nonlinear and linearized one are different.

Limit Cycle

- A system oscillates when it has a nontrivial periodic solution
$$x(t+T) = x(t),\ \forall t \geq 0,\ \text{for some}\, T > 0$$
- The word "nontrivial" is used to exclude the constant solutions.
- The image of a periodic solution in the phase portrait is a closed trajectory, calling periodic orbit or closed orbit.
- We have already seen oscillation of linear system with eigenvalues $\pm j\beta$.
- The origin of the system is a center, and the trajectories are closed
- the solution in Jordan form:
$$z_1(t) = r_0\cos(\beta t + \theta_0), \quad z_2 = r_0\sin(\beta t + \theta_0)$$
$$r_0 = \sqrt{z_{10}^2 + z_{20}^2},\ \theta_0 = \tan^{-1}\frac{z_{20}}{z_{10}}$$
  - r_0: amplitude of oscillation
- Such oscillation where there is a continuum of closed orbits is referred to harmonic oscillator.

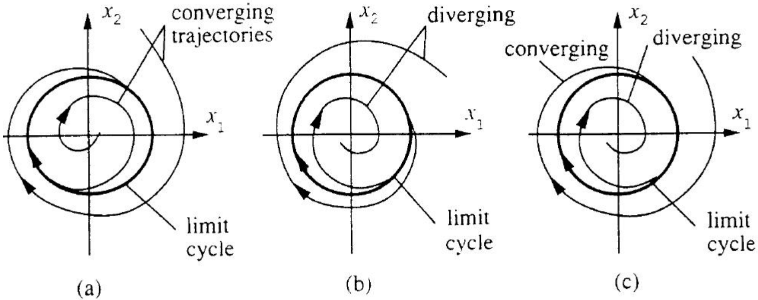
Limit Cycle

- The physical mechanism leading to these oscillations is a periodic exchange of energy stored in the capacitor (electric field) and the inductor (magnetic field).
- We have seen that such oscillation is not robust $\rightsquigarrow$ any small perturbations destroy the oscillation.
- The linear oscillator is not structurally stable
- The amplitude of the oscillation depends on the initial conditions.
- These problems can be eliminated in nonlinear oscillators. A practical nonlinear oscillator can be build such that
  - The nonlinear oscillator is structurally stable
  - The amplitude of oscillation (at steady state) is independent of initial conditions.

Limit Cycle

- On phase plane, a limit cycle is defined as an isolated closed orbit.
- For limit cycle the trajectory should be
  1. closed: indicating the periodic nature of the motion
  2. isolated: indicating limiting nature of the cycle with nearby trajectories converging to/ diverging from it.
- The mass spring damper does not have limit cycle; they are not isolated.
- Depends on trajectories motion pattern in vicinity of limit cycles, there are three type of limit cycle:
  - Stable Limit Cycles: as $t \to \infty$ all trajectories in the vicinity converge to the limit cycle.
  - Unstable Limit Cycles: as $t \to \infty$ all trajectories in the vicinity diverge from the limit cycle.
  - Semi-stable Limit Cycles: as $t \to \infty$ some trajectories in the vicinity converge to/ and some diverge from the limit cycle.

Limit Cycle



Stable, unstable, and semi-stable limit cycles

Example1.a: stable limit cycle

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1)\\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2 - 1)\end{aligned}$$

- Polar coordinates $(x_1 := rcos(\theta),\ x_2 =: rsin(\theta))$

$$\begin{aligned}\dot{r} &= -r(r^2 - 1)\\ \dot{\theta} &= -1\end{aligned}$$

- If trajectories start on the unit circle $(x_1^2(0) + x_2^2(0) = r^2 = 1)$, then $\dot{r} = 0 \Longrightarrow$ The trajectory will circle the origin of the phase plane with period of $\frac{1}{2\pi}$.
- $r < 1 \implies \dot{r} > 0 \Longrightarrow$ trajectories converges to the unit circle from inside.
- $r > 1 \implies \dot{r} < 0 \Longrightarrow$ trajectories converges to the unit circle from outside.
- Unit circle is a stable limit cycle for this system.

Example1.b: unstable limit cycle

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(x_1^2 + x_2^2 - 1)\\ \dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2 - 1)\end{aligned}$$

- Polar coordinates $(x_1 := rcos(\theta),\ x_2 =: rsin(\theta))$

$$\begin{aligned}\dot{r} &= r(r^2 - 1)\\ \dot{\theta} &= -1\end{aligned}$$

- If trajectories start on the unit circle $(x_1^2(0) + x_2^2(0) = r^2 = 1)$, then $\dot{r} = 0 \Longrightarrow$ The trajectory will circle the origin of the phase plane with period of $\frac{1}{2\pi}$.
- $r < 1 \implies \dot{r} < 0 \Longrightarrow$ trajectories diverges from the unit circle from inside.
- $r > 1 \implies \dot{r} > 0 \Longrightarrow$ trajectories diverges from the unit circle from outside.
- Unit circle is an unstable limit cycle for this system.

Example1.c: semi stable limit cycle

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)^2$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)^2$$

- Polar coordinates $(x_1 := rcos(\theta),\ x_2 =: rsin(\theta))$

$$\dot{r} = -r(r^2 - 1)^2$$
$$\dot{\theta} = -1$$

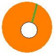
- If trajectories start on the unit circle $(x_1^2(0) + x_2^2(0) = r^2 = 1)$, then $\dot{r} = 0 \Longrightarrow$ The trajectory will circle the origin of the phase plane with period of $\frac{1}{2\pi}$.
- $r < 1 \ \Longrightarrow\ \dot{r} < 0 \Longrightarrow$ trajectories diverges from the unit circle from inside.
- $r > 1 \ \Longrightarrow\ \dot{r} < 0 \Longrightarrow$ trajectories converges to the unit circle from outside.
- Unit circle is a semi-stable limit cycle for this system.

Bendixson's Criterion: Nonexistence Theorem of Limit Cycle

- Gives a sufficient condition for nonexistence of a periodic solution:
- *Suppose Ω is simply connected region in 2-dimention space in this region we define $\nabla f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$. If ∇f is not identically zero over any subregion of Ω and does not change sign in Ω, then Ω contain no limit cycle for the nonlinear system*

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

  - Simply connected set: the boundary of the set is connected + the set is connected
  - Connected set: for connecting any two points belong to the set, there is a line which remains in the set.
  - The boundary of the set is connected if for connecting any two points belong to boundary of the set there is a line which does not cross the set



exmaple of connected but not simply connected set

Bendixson's Criterion

- **Proof by contradiction:**
  - Recall that $\frac{dx_2}{dx_1} = \frac{f_2}{f_1} \implies f_2 dx_1 - f_1 dx_2 = 0$,
  - $\therefore$ Along a closed curve L of a limit cycle:

$$\int_L (f_2 dx_1 - f_1 dx_2) = 0$$

  - Using Stoke's Theorem: ($\int_L f.ndl = \int\int_S \nabla f ds = 0$ S is enclosed by L)

$$\int\int_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 = 0$$

  - This is true if
    - $\nabla f = 0 \; \forall x \in S$ or
    - ∇f changes sign in S
  - This is in contradiction with the assumption that $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ does not vanish and does not change sign $\rightsquigarrow$ there is no closed trajectory.

Nonexistence Theorem of Periodic Solutions for Linear Systems

- Sufficient condition for nonexistence of a periodic solution in linear systems:

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2$$

- $\therefore \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = a_{11} + a_{22} \neq 0 \Longrightarrow$ no periodic sol.
- This is consistent with eigenvalue analysis form of center point which is obtained for periodic solutions:

$$\lambda^2 - (a_{11} + a_{22})\,\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$
$$\text{center} \quad \therefore \quad a_{11} + a_{22} = 0, \quad a_{11}a_{22} - a_{12}a_{21} > 0$$

Limit Cycle

- Example for nonexistence of limit cycle

$$\dot{x}_1 = g(x_2) + 4x_1x_2^2$$
$$\dot{x}_2 = h(x_1) + 4x_1^2x_2$$

- $\therefore \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 4(x_1^2 + x_2^2) > 0 \ \forall x \in \mathcal{R}^2$
- No limit cycle exist in $\mathcal{R}^2$ for this system.
- Note that: there is no equivalent theorem for higher order systems.

Poincare-Bendixson Criterion: Existence Theorem of Limit Cycle

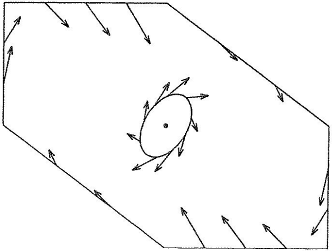
- If there exists a closed and bounded set M s.t.
  1. M contains no equilibrium point or contains only one equilibrium point such that the Jacobian matrix $\frac{\partial f}{\partial x}$ at this point has eigenvalues with positive real parts (unstable focus or node).
  2. Every trajectory starting in M stays in M for all future time

$$\Longrightarrow \quad M \text{ contains a periodic solution}$$

- The idea behind the theorem is that all possible shape of limit points in a plane ($\mathcal{R}^2$) are either equilibrium points or periodic solutions.
- Hence, if the positive limit set contains no equilibrium point, it must have a periodic solution.

Poincare-Bendixson Criterion: Existence Theorem of Limit Cycle

- If M has unstable node/focus, in vicinity of that equilibrium point all trajectories move away
- By excluding the vicinity of unstable node/focus, the set M is free of equilibrium and all trajectories are trapped in it.
- No equivalent theorem for $\mathcal{R}^n, \; n \geq 3$.
- A solution could be bounded in $\mathcal{R}^3$, but neither it is periodic nor it tends to a periodic solution.

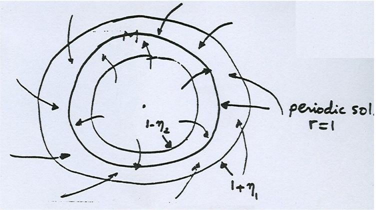


Example for Existence Theorem of Limit Cycle

$$\dot{x}_1 = -x_2 - x_1(x_1^2 + x_2^2 - 1)$$
$$\dot{x}_2 = x_1 - x_2(x_1^2 + x_2^2 - 1)$$

- Polar coordinate:

$$\dot{r} = (1 - r^2)r$$
$$\dot{\theta} = -1$$



  - $\dot{r} \leq 0$ for $r \geq 1 + \eta_1, \ \eta_1 > 0$
  - $\dot{r} \geq 0$for $r \leq 1 - \eta_2, \ 1 > \eta_2 > 0$
- The area found by the circles with radius $1 - \eta_2$ and $1 + \eta_1$ satisfies the condition of the P.B. theorem $\rightsquigarrow$ a periodic solution exists.

Existence Theorem of Limit Cycle

- A method to investigate whether or not trajectories remain inside M:
  - Consider a simple closed curve $V(x) = c$, where $V(x)$ is a continuously differentiable function
  - The vector f at a point x on the curve points
    - inward if the inner product of f and the gradient vector $\nabla V(x)$ is negative:

$$f(x).\nabla V(x) = \frac{\partial V}{\partial x_1} f_1(x) + \frac{\partial V}{\partial x_2} f_2(x) < 0$$

    - outward if $f(x).\nabla V(x) > 0$
    - tangent to the curve if $f(x).\nabla V(x) = 0$.
  - Trajectories can leave a set only if the vector filed points outward at some points on the boundary.
  - For a set of the form $M = \{V(x) \leq c\}$, for some $c > 0$, trajectories trapped inside M if $f(x).\nabla V(x) \leq 0$ on the boundary of M.
  - For an annular region of the form $M = \{W(x) \geq c_1 \text{ and } V(x) \leq c_2\}$ for some $c_1, c_2 > 0$, trajectories remain inside M if $f(x).\nabla V(x) \leq 0$ on $V(x) = c_2$ and $f(x).\nabla W(x) \geq 0$ on $W(x) = c_1$.

Example for Existence Theorem of Limit Cycle

- Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}$$

- The system has a unique equilibrium point at the origin.
- The Jacobian matrix

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 1 - 3x_1^2 - x_2^2 & 1 - 2x_1x_2 \\ -2 - 2x_1x_2 & 1 - x_1^2 - 3x_2^2 \end{bmatrix}_{x=0} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

- has eigenvalues at $1 \pm j\sqrt{2}$
- Let $M = V(x) \leq c$, where $V(x) = x_1^2 + x_2^2$.
- M is bounded and contains eigenvalues with positive real parts

Example for Existence Theorem of Limit Cycle

- On the surface $V(x) = c$, we have

$$\frac{\partial V}{\partial x_1} f_1(x) + \frac{\partial V}{\partial x_2} f_2(x) = 2x_1[x_1 + x_2 - x_1(x_1^2 + x_2^2)]$$
$$+2x_2[-2x_1 + x_2 - x_2(x_1^2 + x_2^2)]$$
$$= 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 - 2x_1x_2$$
$$\leq 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 + (x_1^2 + x_2^2) = 3c - 2c^2$$

- Choosing $c > 1.5$ ensures that all trajectories trapped inside M.
- $\therefore$ by PB criterion, there exits at least one periodic orbit.

Index Theorem

- Let C be a simple closed curve not passing through any equilibrium point
  - Consider the orientation of the vector field $f(\mathrm{x})$ at a point $p \in C$.
  - Let p traverse C counterclockwise $\rightsquigarrow$ the vector $f(x)$ rotates continuously with an angel of $2\pi k$
  - The angle is measured counterclockwise.
  - If C is chosen to encircle a single isolated Equ. point X
  - The integer k is called the index of X
- $k = +1$ for a node or focus, $k = -1$ for saddle.
- Inside any periodic orbit γ, there must be at least one equilibrium point. Suppose the equilibrium points inside γ are hyperbolic, then if N is the number of nodes and foci and S is the number of saddles, it must be that $N - S = 1$.
  - An equilibrium point is hyperbolic if the jacobian at that point has no eigenvalue on the imaginary axis.
  - If the Equ point is not hyperbolic, then k may differ from ± 1
  - It is useful in ruling out the existence of periodic orbits

Index Theorem

- **Example:** The system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1x_2 \\ \dot{x}_2 &= x_1 + x_2 - 2x_1x_2\end{aligned}$$

has two equilibrium points at $(0,0)$ and $(1,1)$. The Jacobian:

$$\left[\frac{\partial f}{\partial x}\right]\Bigg|_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}; \quad \left[\frac{\partial f}{\partial x}\right]\Bigg|_{(1,1)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

- $(0,0)$ is a saddle point and $(1,1)$ is a stable focus.
- The only combination of Equ. points that can be encircled by a periodic orbit is a single focus.
- Periodic orbit in other region such as that encircling both Equ. points are ruled out.