Computational Intelligence
Lecture 2:Fuzzy Sets

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Classical Set

- A classical (crisp) set A in the universe of discourse U: can be defined by
  - List method: listing all of its members
  - Rule method: specifying the properties that must be satisfied by the members of the set

$$A = \{x \in U | x \text{ meets some conditions}\}$$

  - Membership method: introduces a zero-one membership function (also called characteristic function, discrimination function, or indicator function)

$$\mu_A = \begin{cases} 1 & \text{if} x \in A \\ 0 & \text{if} x \notin A \end{cases}$$

Example: cars in Tehran

- The universe of discourse U.
- Set A is the cars with 4 cylinders:

$$A = \{x \in U | x \text{has 4 cylinders}\} \quad OR$$

$$\mu_A = \begin{cases} 1 & \text{if} x \in U \& x \text{ has 4 cylinders} \\ 0 & \text{if} x \in U \& x \text{ does not have 4 cylinder} \end{cases}$$

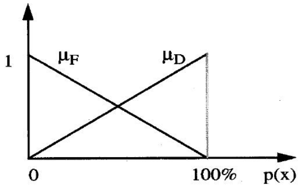
- Set D is the car made in Iran
- BUT the distinction between an Iranian car and a non-Iranian a car is not crisp:(
  - Most of them are not completely made in Iran
- So what should we do??!!

Fuzzy Set

- some sets do not have clear boundaries.
- **Fuzzy set:** in a universe of discourse U is characterized by a membership function $\mu_A(x)$ that takes values in the interval $[0, 1]$.
- In classical sets the membership function of a classical set can only take zero and one
- In fuzzy set the membership function is a continuous function with range $[0, 1]$.
- A fuzzy set A in U is represented by:
  - a set of ordered pairs of a generic element x and its membership value: $A = \{(x, \mu_A(x))|x \in U\}$
  - for continuous U: $A = \int_U \mu_A(x)/x$.
  - for discrete U: $\mu_A(x)$: $A = \sum_U \mu_A(x)/x$
  - $\int$ and $\sum$ do not represent integral and summation.
  - They denote collection of all points $x \in U$ with the associated membership function $\mu_A(x)$

Example: cars in Tehran (Cont'd)

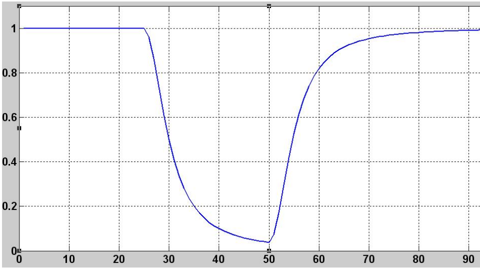
- D: The set "Iranian cars in Iran,"
- $\mu_D = p(x)$
  - $p(x)$ is the percentage of the parts of car x made in the Iran
  - it takes values from 0% to 100%.
- F: The set "non-Iranian cars in Iran,"
- $\mu_F(x) = 1 - p(x)$



- Different membership functions can be defined to characterize the same description.
- The membership functions are not fuzzy, themselves.
- They are precise mathematical functions.
- Fuzzy sets are used to defuzzify the world.
- How to determine the membership functions?
  - Formulate human knowledge
    - Usually, gives a rough formula of the membership function
    - fine-tuning is required.
  - Data collected from various sensors
    - specify the structures of the membership functions and then fine-tune the parameters based on the data.
- A fuzzy set has a one-to-one correspondence with its membership function

Example: Old and Young [1]

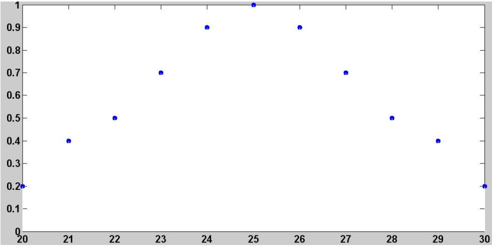
- U is in the interval of $[0, 100]$
- $young = \int_0^{25} 1/x + \int_{25}^{100} (1 + (\frac{x-25}{5})^2)^{-1}/x$
- $old = \int_{50}^{100} (1 + (\frac{x-50}{5})^{-2})^{-1}/x$



Example: A Digital Thermometer

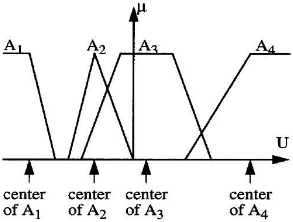
- T: the set for desirable temperature
- $U \in [18, 33]$

$$\mu_T = \frac{.2}{20} + \frac{.4}{21} + \frac{.5}{22} + \frac{.7}{23} + \frac{.9}{24} + \frac{1}{25} + \frac{.9}{26} + \frac{.7}{27} + \frac{.5}{28} + \frac{.4}{29} + \frac{.2}{30}$$



Basic Concepts in Fuzzy Sets

- **Support of a fuzzy set A** in the universe of discourse U is a crisp set that contains all the elements of U that have nonzero membership values in A: $supp_A = \{x \in U | \mu_A > 0\}$
  - In the digital thermometer example: $supp_A = [21, 30]$
  - empty fuzzy set: support is empty
  - fuzzy singleton: support is a single point
- **Center of a fuzzy set:**
  - If the mean value of all points at which the membership function of the fuzzy set achieves its maximum value is finite, then this mean value is the center
  - If the mean value equals positive (negative) infinite, then the center is the smallest (largest) among all points that achieve the maximum membership value.



- **Crossover point of a fuzzy set**: the point in U whose membership value in A equals 0.5.
- **Height of a fuzzy set**: the largest membership value attained by any point.
  - Normal fuzzy set: the height of fuzzy set equals to one (digital thermometer).
- **α-cut of a fuzzy set A** a crisp set A_α contains all the elements in U that have membership values in A greater than or equal to α: $A_\alpha = \{x \in U | \mu_A(x) \geq \alpha\}$
  - In digital thermometer for $\alpha = 0.7, \;\; T_\alpha = [23, 27]$
  - A fuzzy set A is convex iff its α-cut is a convex set for $\forall \alpha \in (0, 1]$.

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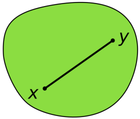
  - A fuzzy set A is convex iff its α-cut is a convex set for $\forall \alpha \in (0, 1]$.
    - In Euclidean space, an object is convex if for every pair of points within the object, every point on the straight line segment that joins them is also within the object.



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  - A fuzzy set A is convex iff its α-cut is a convex set for $\forall \alpha \in (0, 1]$.
    - Let C be a set in a real or complex vector space. C is convex if, $\forall x, y \in C$ and all $\lambda \in [0, 1] \rightsquigarrow, \lambda x + (1 - \lambda) y \in C$

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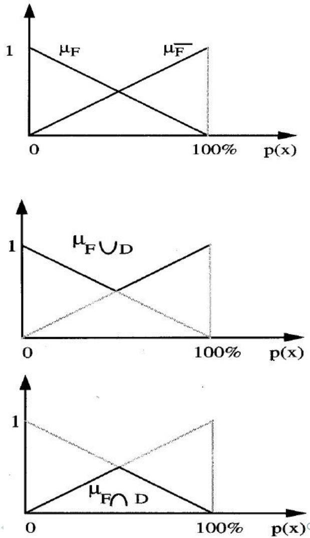
  - In digital thermometer for $\alpha = 0.7,\ \ T_\alpha = [23, 27]$
  - A fuzzy set A is convex iff its α-cut is a convex set for $\forall \alpha \in (0, 1]$.
  - **Lemma:** A fuzzy set $A \in \mathcal{R}^n$ is convex iff $\mu_A[\lambda x_1 + (1 - \lambda)x_2] \geq \min[\mu_A(x_1), \mu_A(x_2)]\ \forall x_1, x_2 \in \mathcal{R}^n, \lambda \in [0, 1]$.

Operations on Fuzzy Sets

- Sets F and D are **equal** iff $\mu_F(x) = \mu_D(x), \forall x \in U$
- Set D **contains** set F ($F \subset D$), iff $\mu_F(x) \leq \mu_D(x), \forall x \in U$
- **Complement of** F is a fuzzy set $\bar{F} \in U$ whose membership function is $\mu_{\bar{F}}(x) = 1 - \mu_F(x)$
- **Union** of sets F and D ($F \cup D$) is a fuzzy set in U: $\mu_{F \cup D} = \max[\mu_F(x), \mu_D(x)]$
  - $F \cup D$ is the smallest fuzzy set containing both F and D.
- **Intersection** of F and D ($F \cap D$) is a fuzzy set in U:$\mu_{F \cap D} = \min[\mu_F(x), \mu_D(x)]$
  - $F \cap D$ is the smallest fuzzy set contained by F and D.



- The De Morgan's Laws are true for fuzzy sets:

$$\overline{F \cup D} = \bar{F} \cap \bar{D}$$
$$\overline{F \cap D} = \bar{F} \cup \bar{D}$$

- For Iranian Cars example:
  - $\mu_{F \cup D} = \begin{cases} \mu_D & if\ 0 \leq p(x) \leq 0.5 \\ \mu_F & if\ 0.5 \leq p(x) \leq 1 \end{cases}$
  - $\mu_{F \cap B} = \begin{cases} \mu_F & if\ 0 \leq p(x) \leq 0.5 \\ \mu_D & if\ 0.5 \leq p(x) \leq 1 \end{cases}$

Further Operations

- An other difference between fuzzy sets and crisp sets:
  - for crisp sets only one type of operation is defined for complement, union, and intersection
  - for fuzzy sets, we can define several operations for them based on the given axioms.
- Why do we need different type of operations?
  - Some operations may not be satisfactory in some situations.

Fuzzy Complement

- Let $c : [0,1] \rightarrow [0,1]$ be a mapping that transforms the membership function of fuzzy set A into the membership function of the complement of A: $c[\mu_A(x)] = \mu_{\bar{A}}(x)$
- It was defined: $c[\mu_A(x)] = 1 - \mu_A$
- Let $a = \mu_A(x)$ and $b = \mu_B(x)$
- the function c is qualified as a complement if:
  - **Axiom c1:** $c(0) = 1$ and $c(1) = 0$ (boundary condition)
  - **Axiom c2:** $\forall a, b \in [0,1]$, if $a < b$, then $c(a) \geq c(b)$ (nonincreasing condition)
    - an increase in membership value must result in a decrease or no change in membership value for the complement

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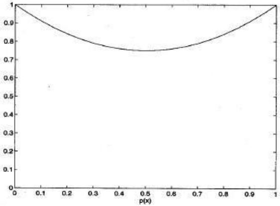
- Some types of fuzzy complement:
  - Sugeno class: $c_\lambda(a) = \frac{1-a}{1+\lambda a}, \quad \lambda \in (-1, \infty)$
    - $\lambda = 0 \rightsquigarrow$ basic fuzzy complement
  - Yager class: $c_w(a) = (1 - a^w)^{1/w}, \quad w \in (0, \infty)$
    - $w = 1 \rightsquigarrow$ basic fuzzy complement

Fuzzy set-S Norm

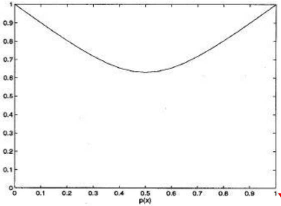
- Let $s : [0,1] \times [0,1] \rightarrow [0,1]$ be a mapping that transforms the membership functions of fuzzy sets A and B into the membership function of the union of A and B, that $s[\mu_A(x), \mu_B(x)] = \mu_{A \bigcup B}$.
- the function S to be qualified as an union
  - **Axiom s1**.$s(1,1) = 1, s(0,a) = s(a,O) = a$ (boundary condition).
  - **Axiom s2**. $s(a,b) = s(b,a)$ (commutative condition).
  - **Axiom s3**. If $a \leq a'$ and $b \leq b'$, then $s(a,b) \leq s(a',b')$ (nondecreasing condition).
  - **Axiom s4**. $s(s(a,b),c) = s(a,s(b,c))$ (associative condition).
- Popular types of s-norm
  - Dombi class: $s_\lambda(a,b) = \frac{1}{1+[(\frac{1}{a}-1)^{-\lambda}+(\frac{1}{b}-1)^{-\lambda}]^{-1/\lambda}}, \ \lambda \in (0,\infty)$
  - Dobios-Prade class: $s_\alpha(a,b) = \frac{a+b-ab-\min(a,b,1-\alpha)}{\max(1-a,1-b,\alpha)}, \alpha \in [0,1]$
  - Yager class: $s_w(a,b) = min[1,(a^w + b^w)^{1/w}], \ w \in (0,\infty)$

- Other type of s-norm
  - Drastic sum: $s_{ds}(a,b) = \begin{cases} a \text{ if } b = 0 \\ b \text{ if } a = 0 \\ 1 \text{ otherwise} \end{cases}$
  - Einstein sum: $s_{es}(a,b) = \frac{a+b}{1+ab}$
  - Algebric sum: $s_{as}(a,b) = a + b - ab$
- **Theorem:** *For any s-norm s, that is for any function $s : [0,1] \times [0,1] \to [0,1]$ that satisfies Axioms s1-s4, the smallest s-norm is maximum and the largest is drastic s-norm*
- Proof:
  - Axioms s1 and s3 $\Rightarrow s(a,b) \geq s(a,0) = a$
  - Axiom s2 $\Rightarrow s(a,b) = s(b,a) \geq s(b,0) = b$
  - $\therefore s(a,b) \geq max(a,b)$
  - If $b = 0$, Axiom s1 $\Rightarrow s(a,b) = s(a,0) = a \leadsto s(a,b) = s_{ds}(a,b)$
  - If $a = 0$, Axiom s2 $\Rightarrow s(a,b) = s_{ds}(a,b)$
  - If $a \neq 0 \& b \neq 0$, $s_{ds}(a,b) = 1 \geq s(a,b)$
  - $\therefore s(a,b) \leq s_{ds}(a,b), \forall a, b \in [0,1]$

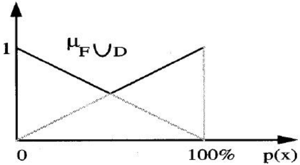
- Example: The Iranian cars
  - Using Algebric sum:
    $\mu_{F \cup D} = p(x) + (1 - p(x)) - p(x)(1 - p(x)) = 1 - p(x) + p(x)^2$
  - Using Yager s-norm, $w = 3$:
    $\mu_{F \cup D} = \min[1, (p(x)^3 + (1 - p(x))^3)^{1/3}]$



Algebric



Yager



max

Classical Set

- **Lemma 1:** For Dombi class s-norm and Drastic class s-norm it can be defined

$$\lim_{\lambda\to\infty} s_\lambda(a,b) = max(a,b)$$
$$\lim_{\lambda\to 0} s_\lambda(a,b) = s_{ds}(a,b)$$

- **Lemma 2:** For Yager class s-norm and Drastic class s-norm it can be defined

$$\lim_{w\to\infty} s_w(a,b) = max(a,b)$$
$$\lim_{w\to 0} s_w(a,b) = s_{ds}(a,b)$$

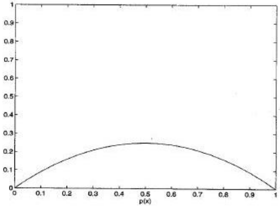
Fuzzy Intersection- T-Norm

- Let $t : [0,1] \times [0,1] \rightarrow [0,1]$ be a mapping that transforms the membership functions of fuzzy sets A and B into the membership function of the union of A and B, that $t[\mu_A(x), \mu_B(x)] = \mu_{A \cap B}$.
- the function T to be qualified as an intersection
  - **Axiom t1**. $t(0,0) = 0, t(0,a) = t(a,0) = a$ (boundary condition).
  - **Axiom t2**. $t(a,b) = t(b,a)$ (commutative condition).
  - **Axiom t3**. If $a \leq a'$ and $b \leq b'$, then $t(a,b) \leq t(a',b')$ (nondecreasing condition).
  - **Axiom t4**. $t(t(a,b),c) = t(a,t(b,c))$ (associative condition).
- Popular types of t-norm
  - Dombi class: $t_\lambda(a,b) = \frac{1}{1+[(\frac{1}{a}-1)^\lambda+(\frac{1}{b}-1)^\lambda]^{1/\lambda}}, \ \lambda \in (0,\infty)$
  - Dobios-Prade class: $t_\alpha(a,b) = \frac{ab}{\max(a,b,\alpha)}, \alpha \in [0,1]$
  - Yager class:
    $t_w(a,b) = 1 - min[1, ((1-a)^w + (1-b)^w)^{1/w}], \ w \in (0,\infty)$

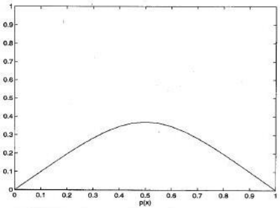
- Other type of t-norm
  - Drastic product: $t_{ds}(a,b) = \begin{cases} a \text{ if } b = 1 \\ b \text{ if } a = 1 \\ 0 \text{ otherwise} \end{cases}$
  - Einstein product: $t_{ep}(a,b) = \frac{ab}{2-(a+b-ab)}$
  - Algebric product: $t_{ap}(a,b) = ab$
- **Theorem:** *For any t-norm t, that is for any function $t : [0,1] \times [0,1] \rightarrow [0,1]$ that satisfies Axioms t1-t4, the smallest t-norm is minimum and the largest is drastic t-norm*
- prove it.
- **Lemma 3:** For Dombi class t-norm and Drastic class t-norm it can be defined

$$\lim_{\lambda \to \infty} t_\lambda(a,b) = min(a,b)$$
$$\lim_{\lambda \to 0} t_\lambda(a,b) = t_{dp}(a,b)$$

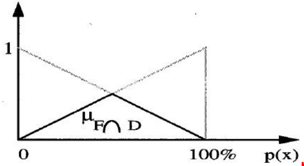
- Example: One more time, the Iranian cars
  - Using Algebric product: $\mu_{F \cap D} = p(x)(1 - p(x))$
  - Using Yager t-norm, $w = 3$: $\mu_{F \cup D} = 1 - \min[1, ((1 - p(x))^3 + p(x)^3)^{1/3}]$



Algebric



Yager

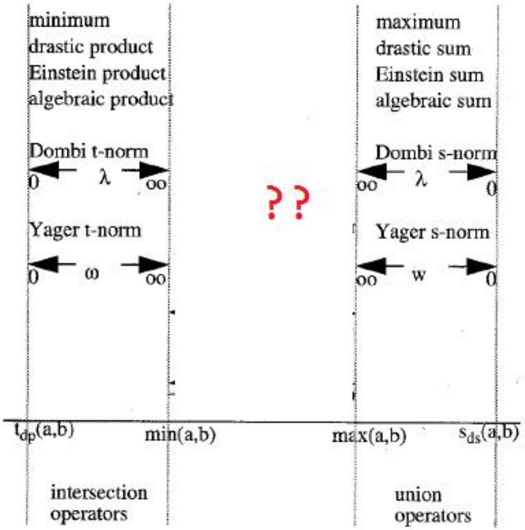


min

- If the s-norm $s(a,b)$, t-norm $t(a,b)$ and fuzzy complement $c(a)$ satisfy the following equation, they form an associated class (DeMorgan's Law)

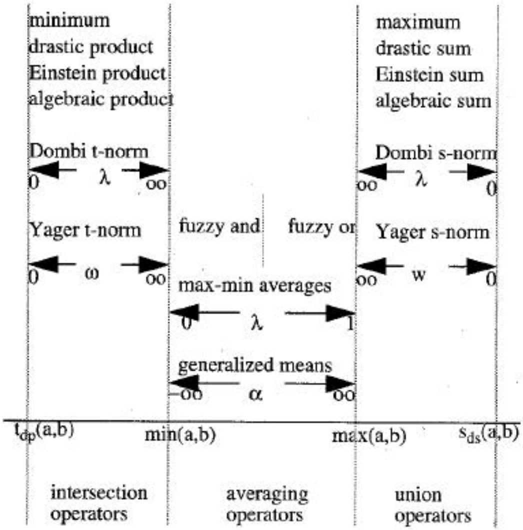
$$c[s(a,b)] = t[c(a), c(b)]$$

- Example: Show that the Yager s-norm and t-norm with the basic complement are associated
  - $c[s_w(a,b)] = 1 - \min[1, (a^w + b^w)^{1/w}]$
  - $t_w[c(a), c(b)] = 1 - \min[1, ((1 - 1 + a)^w + (1 - 1 + b)^w)^{1/w}]$


minimum
drastic product
Einstein product
algebraic product
Dombi t-norm
0 λ oo
Yager t-norm
0 ω oo
maximum
drastic sum
Einstein sum
algebraic sum
Dombi s-norm
oo λ 0
Yager s-norm
oo w 0
??
t_dp(a,b)
min(a,b)
max(a,b)
s_ds(a,b)
intersection operators
union operators

Averaging Operator

- This operator fills the gap between $\min(a, b)$, and $\max(a, b)$
- Some average operators:
  - Max-min average: $v_\lambda(a, b) = \lambda \max(a, b) + (1 - \lambda) \min(a, b), \;\lambda \in [0, 1]$
  - Generalized means: $v_\alpha(a, b) = {\frac{a^\alpha + b^\alpha}{2}}^{1/\alpha}, \quad \alpha \in R, \;\alpha \neq 0$
  - Fuzzy and: $v_p(a, b) = pmin(a, b) + \frac{(1-p)(a+b)}{2}, \;p =\in [0, 1]$
  - Fuzzy or: $v_\gamma(a, b) = \gamma \max(a, b) + \frac{(1-\gamma)(a+b)}{2}, \;\gamma \in [0, 1]$



Full Scope of Fuzzy Operators

L. A. Zadeh, “Fuzzy sets,” *Informat. Control*, vol. 8, , pp. 338–353, 1965.