# Computational Intelligence
# Lecture 16: Fuzzy Systems as Nonlinear Mapping

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Some Classes of Fuzzy Systems

Fuzzy Systems as Universal Approximators

Design of a Fuzzy Approximator

- Among 45 types of fuzzy systems obtained by combining different types of inference engines (5), fuzzifiers (3), and defuzzifiers (3) just some of them are useful.
- Fuzzy Systems with Center Average Defuzzifier
  - Suppose that the output fuzzy set $B^l$ is normal with center $\bar{y}^l$. Then the fuzzy systems with
    - canonical fuzzy rule base,
    - product inference engine,
    - singleton fuzzifier,
    - center average defuzzifier

    are of the following form:

$$f(x) = \frac{\sum_{l=1}^{M} \bar{y}^l (\prod_{i=1}^{n} \mu_{A_i^l}(x_i^*))}{\sum_{l=1}^{M} (\prod_{i=1}^{n} \mu_{A_i^l}(x_i^*))} \tag{1}$$

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    where $x \in U \subset R^n$ is input; $f(x) = y^* \in V \subset R$ is output
    - Using singleton fuzzifier and product inf. eng. $\mu_{B'}(y) = \max_{l=1}^{M}[\prod_{i=1}^{n} \mu_{A_i^l}(x_i^*)\mu_{B^l}(y)]$
    - Assume for $l$th rule, $\bar{y}^l$ is the center, since $B^l$ is normal, height of $\bar{y}^l$ is $\prod_{i=1}^{n} \mu_{A_i^l}(x_i^*)\mu_{B^l}(y) = \prod_{i=1}^{n} \mu_{A_i^l}(x_i^*)$
    - Define $y^* = f(x)$ and using center of average defuzzifier yields (1)

- Using (1), one can provide a systematic procedure for transforming a set of linguistic rules into a nonlinear mapping.
- The most popular mem. fcn of $\mu_{A_i^l}, \mu_{B^l}$ is Gaussian:

  $\mu_{A_i^l}(x_i) = a_i^l exp[-(\frac{x_i - \bar{x}_i^l}{\sigma_i^l})^2]$

  $\mu_{B^l}(y) = exp[-(y - \bar{y}^l)^2]$

-

$$\therefore f(x) = \frac{\sum_{l=1}^{M} \bar{y}^l (\prod_{i=1}^{n} a_i^l exp[-(\frac{x_i - \bar{x}_i^l}{\sigma_i^l})^2])}{\sum_{l=1}^{M} (\prod_{i=1}^{n} a_i^l exp[-(\frac{x_i - \bar{x}_i^l}{\sigma_i^l})^2])} \quad (2)$$

- Other choices could be triangular and trapezoid mem. fcn.

- Another class of commonly used fuzzy systems is obtained by replacing the product inference engine with the minimum inference engine.
- A Fuzzy system with canonical fuzzy rule base
  - minimum inference engine
  - singleton fuzzifier
  - center average defuzzifier

  are of the following form:

$$f(x) = \frac{\sum_{l=1}^{M} \bar{y}^l (\min_{i=1}^{n} \mu_{A_i^l}(x_i^*))}{\sum_{l=1}^{M} (\min_{i=1}^{n} \mu_{A_i^l}(x_i^*))} \tag{3}$$

- Since $Sup_{y \in V}$ and min are not always interchangeable, obtaining closed-form formulas for fuzzy systems with maximum defuzzifier and Lukasiewicz, Zadeh, or Dienes-Rescher inference engines is difficult
- For such systems the output of the fuzzy system has to be computed in a step-by-step fashion:
  - computing the outputs of fuzzifier,
  - fuzzy inference engine,
  - and then defuzzifier

# Fuzzy Systems as Universal Approximators

- We are interested to investigate the capability of fuzzy systems as a function approximator
- It is useful in controllers, decision makers, signal processors and etc.
- **Universal Approximation Theorem** Suppose that the input universe of discourse $U$ is a compact set in $R^n$: Then, for any given real continuous function $g(x)$ on $U$ and arbitrary $\epsilon > 0$, there exists a fuzzy system $f(x)$ in the form of (2) s.t. $\sup_{x \in U} |f(x) - g(x)| < \epsilon$
- **Proof:** is Obtained using Stone-Weierstrass Theorem,
- The fuzzy systems with product inference engine, singleton fuzzifier, center average defuzzifier, and Gaussian membership functions are universal approximators.

- The universal approximator theorem guarantees only existence a fuzzy system capable in approximating any function to arbitrary accuracy.
- For engineers just knowing the existence is not enough
- They should develop such fuzzy system!!

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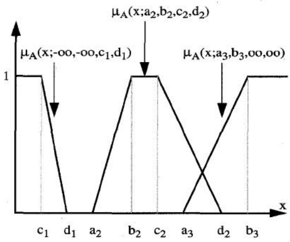
  3. The analytic formula of $g(x)$ is unknown and we are provided only a limited number of I/O $(x_j, g(x_j))$ , where $x_j \in U$ cannot be arbitrarily chosen.
     - especially true for fuzzy control when due to stability requirements arbitrary input values may not be possible

# Preliminary Definitions

- **Pseudo-Trapezoid Membership Function** Let $[a, d] \subset R$. The pseudo-trapezoid membership function of fuzzy set $A$ is a continuous function in $R$ given by

$$\mu_A(x; a, b, c, d, H) = \begin{cases} I(x) & x \in [a, b) \\ H & x[b, c] \\ D(x) & x \in (c, d] \\ 0 & R - (a, b) \end{cases}$$

  where
  - $a \leq b \leq c \leq d$
  - $0 < H \leq 1; H = 1$ if $A$ is normal
  - $0 \leq I(x) \leq 1$
  - $0 \leq D(x) \leq 1$ a nondecreasing function in $[c, d)$




- **Example:** $a = d = \infty, b = c = \bar{x}$
  $I(x) = D(x) = exp(-(\frac{x-\bar{x}}{\sigma})^2)$
  - $\therefore$ They become Gaussian membership functions

- **Completeness of Fuzzy Sets**: Fuzzy sets $A^1, A^2, ..., A^N \in W \subset R$ are said to be complete on $W$ if for any $x \in W$, there exists $A_i$ s.t. $\mu_{A_i}(x) > 0$.

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- **High Set of Fuzzy Set**. The high set of a fuzzy set $A \in W \subset R$ is a subset in $W$: $hgh(A) = \{x \in W | \mu_A(x) = \sup_{x' \in W} \mu_A(x')\}$
  - If $A$ is a normal fuzzy set with pseudo-trapezoid mem. fcn. $hgh(A) = [b, c]$.

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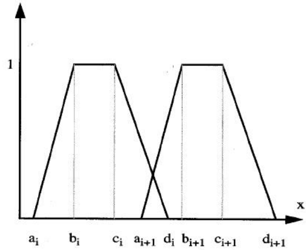
- **Order Between Fuzzy Sets** For two fuzzy sets $A$ and $B \in W \subset R$, $A > B$ if $hgh(A) > hgh(B)$
  - i.e., $x \in hgh(A)$ and $x' \in hgh(B) \rightsquigarrow x > x'$

- **Lemma:** If $A^1, A^2, ..., A^N$ are consistent and normal fuzzy sets in $W \subset R$ with pseudo-trapezoid membership functions $\mu_{A^i}(x; a_i, b_i, c_i, d_i)(i = 1, 2, ..., N)$, then there exists a rearrangement $\{i_1, i_2, ..., i_N\}$ of $\{1, 2, ..., N\}$ s.t.
  $A^{i1} < A^{i2} < ... < A^{iN}$
  - If $A^1 < A^2 < ... < A^N$, then
    $c_i \leq a_{i+1} < d_i \leq b_{i+1}$ for $i = 1, 2, ..., N-1$.



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1. Define fuzzy sets $A_i^1, A_i^2, ..., A_i^{N_i}, \ \ i = 1, 2$ in $[\alpha_i, , \beta_i]$ which are normal, consistent, complete with pesudo-trapezoid membership functions
   - $\mu_{A_i^1}(x_1, a_i^1, b_i^1, c_i^1, d_i^1), \ldots,$ $\mu_{A_i^{N_i}}(x_i, a_i^{N_i}, b_i^{N_i}, c_i^{N_i}, d_i^{N_1})$
   - $A_i^1 < A_i^2 < .... < A_i^{N_i}$
   - $a_i^1 = b_i^1 = \alpha_i$ and $c_i^{N_i} = d_i^{N_i} = \beta_i$
   - Define $e_1^1 = \alpha_1, e_1^{N_1} = \beta_1$, $e_1^j = \frac{1}{2}(b_1^j + c_1^j)$ for $j = 2, ..., N_1 - 1$
   - $e_2^1 = \alpha_2, e_2^{N_2} = \beta_2, \ e_2^j = \frac{1}{2}(b_2^j + c_2^j)$ for $j = 2, ..., N_2 - 1$

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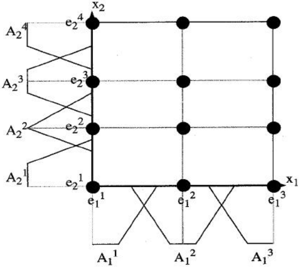
E.g.: $N_1 = 3, N_2 = 4,$ $\alpha_1 = \alpha_2 = 0,$ $\beta_1 = \beta_2 = 1$



2. Construct $M = N_1 \times N_2$ fuzzy IF-THEN rules:
   $Ru^{i_1 i_2}$ : IF $x_1$ is $A_1^{i1}$ and $x_2$ is $A_2^{i2}$, THEN $y$ is $B^{i_1 i_2}$
   - $i_1 = 1, 2, ..., N_1,\ i_2 = 1, ..., N_2$
   - The center of fuzzy set $B^{i_1 i_2}$ is $\bar{y}^{i_1 i_2} = g(e_1^{i1}, e_2^{i2})$
   - In E.g., There are $3 \times 4 = 12$ rules; $\bar{y}^{i_1 i_2}$ are the 12 dark points

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3. Construct the fuzzy system $f(x)$ from the $N_1 \times N_2$ rules, using product inference engine , singleton fuzzifier ; and center average defuzzifier

$$f(x) = \frac{\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \bar{y}^{i_1 i_2} (\mu_{A_1}^{i_1}(x_1) \mu_{A_2}^{i_2}(x_2))}{\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} (\mu_{A_1}^{i_1}(x_1) \mu_{A_2}^{i_2}(x_2))}$$

   - The fuzzy sets $A_i^1 ... A_i^{N_i}$ are complete $\rightsquigarrow f(x)$ is well defined