Computational Intelligence
Lecture 15: Multi-Layer Perceptron

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Linearly Nonseparable Pattern Classification

- A single layer network can find a linear discriminant function.
- Nonlinear discriminant functions for linearly nonseparable function can be considered as piecewise linear function
- The piecewise linear discriminant function can be implemented by a multilayer network
- The pattern sets $\dagger_1$ and $\dagger_2$ are linearly nonseparable, if no weight vector w exists s.t

$$y^T w > 0 \text{ for each} y \in \dagger_1$$
$$y^T w < 0 \text{ for each} y \in \dagger_2$$

Example XOR

- XOR is nonseparable

x_1	x_2	Output
1	1	1
1	0	-1
0	1	-1
0	0	1

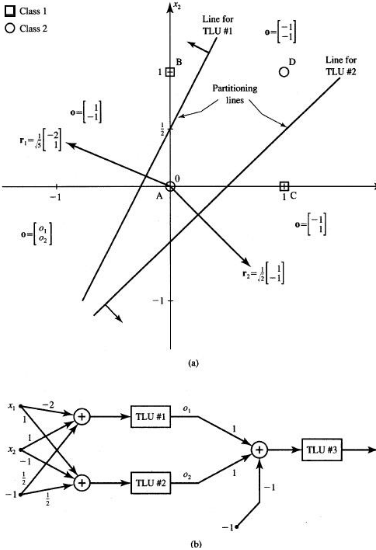
- At least two line are required to separate them
- By choosing proper values of weights, the decision lines are

$$-2x_1 + x_2 - \frac{1}{2} = 0$$

$$x_1 - x_2 - \frac{1}{2} = 0$$

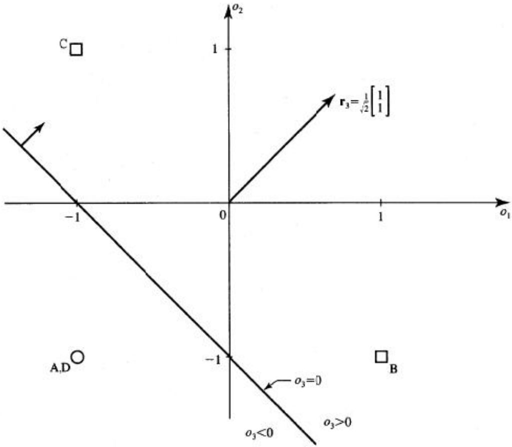
- output of the first layer network:

$$o_1 = sgn(-2x_1 + x_2 - \frac{1}{2}) \quad o_2 = sgn(x_1 - x_2 - \frac{1}{2})$$



Symbol	Pattern Space x_1	Pattern Space x_2	Image Space o_1	Image Space o_2	TLU #3 Input $o_1 + o_2 + 1$	Output Space o_3	Class Number
A	0	0	−1	−1	−	−1	2
B	0	1	1	−1	+	+1	1
C	1	0	−1	1	+	+1	1
D	1	1	−1	−1	−	−1	2

(a)



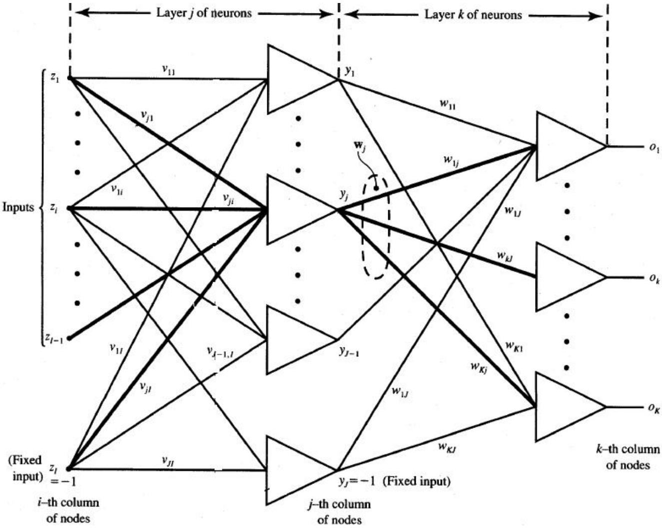
- The main idea of solving linearly nonseparable patterns is:
  - the set of linearly nonseparable pattern is mapped into the image space where is becomes linearly separable.
  - This can be done by proper selecting weights of the first layer(s)
  - Then in the next layer they can be easily classified
- Increasing # of neurons in the middle layer increases # of lines.
  - $\therefore$ provides nonlinear and more complicated discriminant functions
- The pattern parameters and center of clusters are not always known a priori
- $\therefore$ A stepwise supervised learning algorithm is required to calculate the weights

Delta Learning Rule for Feedforward Multilayer Perceptron

- The training algorithm is called error back propagation (EBP) training algorithm
- If a submitted pattern provides an output far from desired value, the weights and thresholds are adjusted s.t. the current mean square classification error is reduced.
- The training is continued/repeated for all patterns until the training set provide an acceptable overall error.
- Usually the mapping error is computed over the full training set.
- EBP alg. is working in two stages:
    1. The trained network operates feedforward to obtain output of the network
    2. The weight adjustment propagate backward from output layer through hidden layer toward input layer.

Multilayer Perceptron

- input vec. z
- output vec. o
- output of first layer, input of hidden layer y
- activation fcn. $\Gamma(.) = diag\{f(.)\}$



Feedforward Recall

- Give training pattern vector z, result of this phase is computing the output vector o (for two layer network)
  - Output of first layer: $y = \Gamma[Vz]$ (the internal mapping $z \rightarrow y$)
  - Output of second layer: $o = \Gamma[Wy]$
  - Therefore:

$$o = \Gamma[W\Gamma[Vz]]$$

- Since the activation function is assumed to be fixed, weights are the only parameters should be adjusted by training to map $z \longrightarrow o$ s.t. o matches d
- The weight matrices W and V should be adjusted s.t. $\|d - o\|^2$ is min.

Back-Propagation Training

- Training is started by feedforward recall phase
- The error signal vector is determined in the output layer
- The error is defined for a single perceptron is generalized to include all squared error at the outputs $k = 1, ..., K$

$$E_p = \frac{1}{2}\Sigma_{k=1}^{K}(d_{pk} - o_{pk})^2 = \frac{1}{2}\|d_p - o_p\|^2$$

  - p: pth pattern
  - d_p: desired output for pth pattern
- Bias is the jth weight corresponding to jth input $y_j = -1$
- Then it propagates toward input layer
- The weights should be updated from output layer to hidden layer

Back-Propagation Training

- Recall the learning rule of continuous perceptron (it is so-called delta learning rule)

$$\Delta w_{kj} = -\eta \frac{\partial E}{\partial w_{kj}}$$

  p is skipped for brevity.

- for each neuron in layer k:

$$net_k = \sum_{j=1}^{J} w_{kj} y_j$$
$$o_k = f(net_k)$$

- Define the error signal term
  $\delta_{ok} = -\frac{\partial E}{\partial(net_k)} = (d_k - o_k) f'(net_k), \;\; k = 1, \ldots, K$
- $\therefore \Delta w_{kj} = -\eta \frac{\partial E}{\partial(net_k)} \frac{\partial(net_k)}{\partial w_{kj}} = \eta \delta_{ok} y_j \;\; for \;\; k = 1, \ldots, K, \;\; j = 1, \ldots, J$

- The weights of output layer w can be updated based in delta rule, since desired output is available for them
- For updating the hidden layer weights:

$$\Delta v_{ji} = -\eta \frac{\partial E}{\partial v_{ji}}$$

$$\frac{\partial E}{\partial v_{ji}} = \frac{\partial E}{\partial net_i} \frac{\partial net_j}{\partial v_{ji}}, i = 1, ..., n \; j = 1, ...n$$

- $net_j = \sum_{i=1}^{I} v_{ji} z_i \rightsquigarrow \frac{\partial net_j}{\partial v_{ji}} = z_i$ which are input of this layer
- where $\delta_{yj} = -\frac{\partial E}{\partial (net_j)}$ for $j = 1, ..., J$ is signal error of hidden layer
- $\therefore$, the hidden layer weights are updated by $\Delta v_{ji} = \eta \delta_{yj} z_i$

- Despite of the output layer where net_k affected the kth neuron output only, net_j contributes to every K terms of error $E = \frac{1}{2}\sum_{k=1}^{R}(d_k - o_k)^2$

$$\delta_{yj} = -\frac{\partial E}{\partial y_j}.\frac{\partial y_j}{\partial net_j}$$

$$\frac{\partial y_j}{\partial net_j} = f'(net_j)$$

$$\frac{\partial E}{\partial y_j} = -\sum_{k=1}^{R}(d_k - o_k)f'(net_k)\frac{\partial net_k}{\partial y_j} = -\sum_{k=1}^{R}\delta_{ok}w_{kj}$$

- $\therefore$ The updating rule is

$$\Delta v_{ji} = \eta f'(net_j)z_i\sum_{k=1}^{R}\delta_{ok}w_{kj} \qquad (1)$$

- So the delta rule for hidden layer is:

$$\Delta v = \eta\delta x \qquad (2)$$

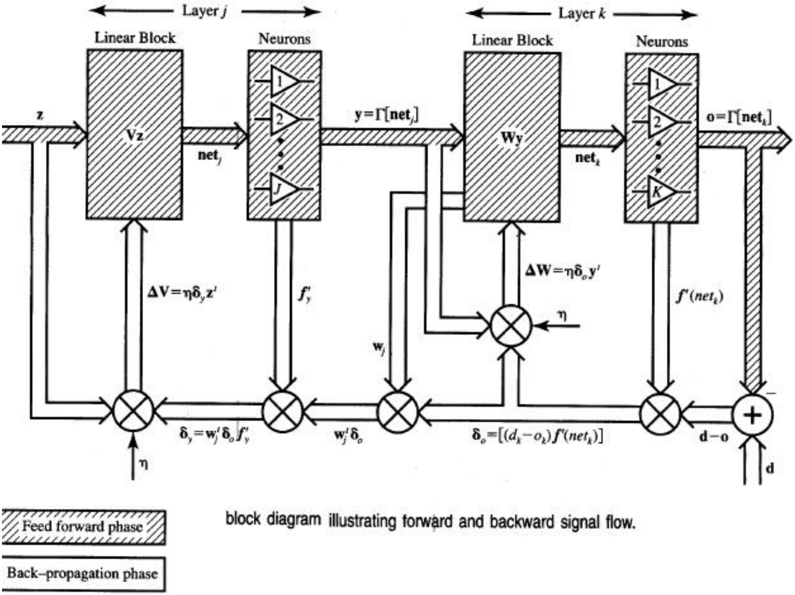
where η is learning const., δ is layer error, and x is layer input.
- The weights of jth layer is proportional to the weighted sum of all δ of next layer.
- Delta training rule of output layer and generalized delta learning rule for hidden layer have fairly uniform formula.
- But
  - $\delta_o = (d_k - o_k)o_k(1 - o_k)$ contains scalar entries, contains error between desired and actual output times derivative of activation function
  - $\delta_y = w_j\delta_o f'y$ contains the weighted sum of contributing error signal δ_o produced by the following layer
  - The learning rule propagates the error back by one layer

Back-Propagation Training

- Assuming sigmoid activation function, its time derivative is

$$f'(net) = \begin{cases} o(1-o) & unipolar: \; f(net) = \frac{1}{1+exp(-\lambda net)}, \lambda = 1 \\ \frac{1}{2}(1-o^2) & bipolar: \; f(net) = \frac{2}{1+exp(-\lambda net)} - 1, \lambda = 1 \end{cases}$$

- After all training patterns are applied, the learning procedure stops when the final error is below the upper bound E_{max}
- In fig next page, the shaded path refers to feedforward path and blank path is Back-Propagation (BP) mode



block diagram illustrating forward and backward signal flow.

Error Back-Propagation Training Algorithm

- Given P training pairs $\{z_1, d_1,\ z_2, d_2,\ ..., z_p, d_p\}$ where z_i is $(I \times 1)$, d_i is $(K \times 1)$, $i = 1, ..., P$
  - The Ith component of each z_i is of value -1 since input vectors are augmented .
- Size $J - 1$ of the hidden layer having outputs y is selected.
  - Jth component of y is -1, since hidden layer have also been augmented.
  - y is $(J \times 1)$ and o is $(K \times 1)$
- In the following, q is training step and p is step counter within training cycle.
  1. Choose $\eta > 0,\ E_{max} > 0$
  2. Initialized weights at small random values, W is $(K \times J)$, V is $(J \times I)$
  3. Initialize counters and error: $q \longleftarrow 1,\ p \longleftarrow 1, E \longleftarrow 0$
  4. Training cycle begins here. Set $z \longleftarrow z_p, d \longleftarrow d_p$,
     $y_j \leftarrow f(v_j^t z),\ j = 1, .., J$ (v_j a column vector, jth row of V)
     $o \leftarrow f(w_k^t y),\ k = 1, ..., K$ (w_k a column vector, kth row of W)(f(net) is sigmoid function)

Error Back-Propagation Training Algorithm Cont'd

5. Find error: $E \longleftarrow \frac{1}{2}(d-o)^2 + E$ for $k = 1, ..., K$
6. Error signal vectors of both layers are computed. δ_o (output layer error) is $K \times 1$, δ_y (hidden layer error) is $J \times 1$
   $\delta_{ok} = \frac{1}{2}(d_k - o_k)(1 - o_k^2)$, for $k = 1, ..., K$
   $\delta_{yj} = \frac{1}{2}(1 - y_j^2)\sum_{k=1}^{K} \delta_{ok} w_{kj}$, for $j = 1, ..., J$
7. Update weights:
   - Output $w_{kj} \longleftarrow w_{kj} + \eta\delta_{ok}y_j$, $k = 1, .., K$ $j = 1, .., J$
   - Hidden layer $v_{ji} \longleftarrow v_{ji} + \eta\delta_{yj}z_j$, $jk = 1, .., J$ $i = 1, .., I$
8. If $p < P$ then $p \longleftarrow p + 1, q \longleftarrow q + 1$, go to step 4, otherwise, go to step 9.
9. If $E < E_{max}$ the training is terminated, otherwise $E \longleftarrow 0, p \longleftarrow 1$ go to step 4 for new training cycle.

Multilayer NN as Universal Approximator

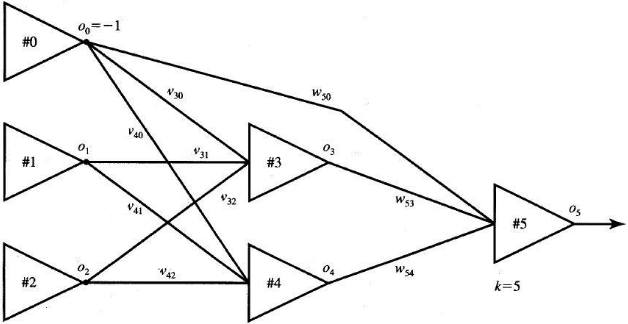
- Although classification is an important application of NN, considering the output of NN as binary response limits the NN potentials.
- We are considering the performance of NN as universal approximators
- Finding an approximation of a multivariable function $h(x)$ is achieved by a supervised training of an input-output mapping from a set of examples
- Learning proceeds as a sequence of iterative weight adjustment until is satisfies min distance criterion from the solution weight vectors w^*.
- Several theorem such as Kolmogorov and Hecht-Nielsen Theorems guarantee existence of an approximating fcn. g.
- No more guideline is provided for finding such functions

- Some other theorems have been given some hints on choosing activation functions (Lee & Kil 1991, Chen 1991, Cybenko 1989)
- **Cybenko Theorem** Let I_n denote the n-dimensional unit cube, $[0,1]^n$. The space of continuous functions on I_n is denoted by $C(I_n)$. Let g be any continuous sigmoidal function of the form
$$g \rightarrow \begin{cases} 1 & as\ t \rightarrow \infty \\ 0 & as\ t \rightarrow -\infty \end{cases}$$
Then the finite sums of the form
$$F(x) = \sum_{i=1}^{N} v_i g(\sum_{j=1}^{n} w_{ij}^T x_j + \theta)$$
are dense in $C(I_n)$. In other words, given any $f \in C(I_n)$ and $\epsilon > 0$, there is a sum $F(x)$ of the above form for which
$$|F(x) - f(x)| < \epsilon \ \ \forall\ x \in I_n$$

- MLP can provide all the conditions of Cybenko theorem
  - θ is bias
  - w_{ij} is weights of input layer
  - v_i is output layer weights
- Failures in approximation can be attribute to
  - Inadequate learning
  - Inadequate $\#$ of hidden neurons
  - Lack of deterministic relationship between the input and target output
- If the function to be approximated is not bounded, there is no guarantee for acceptable approximation

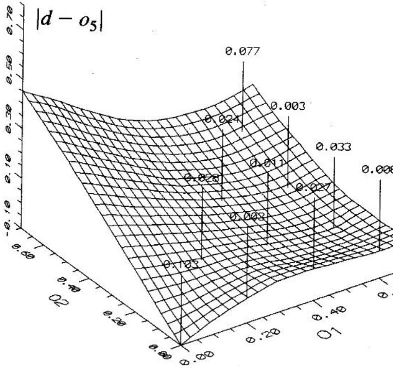
Example

- Consider a three neuron network
- Bipolar activation function
- **Objective:** Estimating a function which computes the length of input vector $d = \sqrt{o_1^2 + o_2^2}$
- $o_5 = \Gamma[W\Gamma[Vo]]$, $o = [-1 \; o_1 \; o_2]$
- Inputs o_1, o_2 are chosen $0 < o_i < 0.7$ for $i = 1, 2$



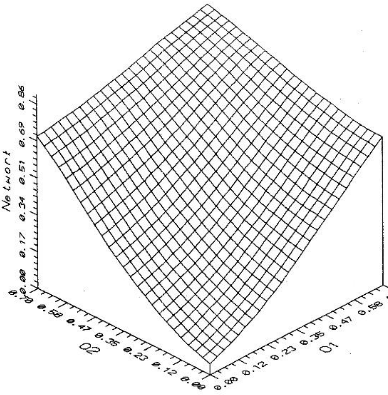
Example Cont'd, Experiment 1

- Using 10 training points which are informally spread in lower half of first plane
- The training is stopped at error 0.01 after 2080 steps
- $\eta = 0.2$
- The weights are $W = [0.03\ 3.66\ 2.73]^T$,
  $$V = \begin{bmatrix} -1.29 & -3.04 & -1.54 \\ 0.97 & 2.61 & 0.52 \end{bmatrix}$$
- Magnitude of error associated with each training pattern are shown on the surface
- Any generalization provided by trained network is questionable.

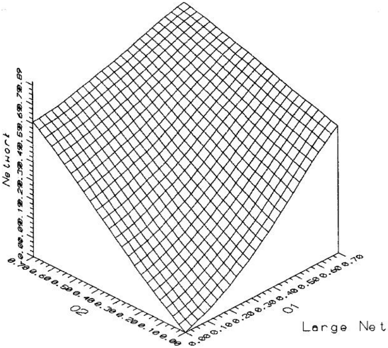


Example Cont'd, Experiment 2

- Using the same architecture but with 64 training points covering the entire domain
- The training is stopped at error 0.02 after 1200 steps
- $\eta = 0.4$
- The weights are
  $W = [-3.74 \ \ -1.8 \ 2.07]^T$,
  $V = \begin{bmatrix} -2.54 & -3.64 & 0.61 \\ 2.76 & 0.07 & 3.83 \end{bmatrix}$
- The mapping is reasonably accurate
- Response at the boundary gets worse.



Example Cont'd, Experiment 3



- Using the same set of training points and a NN with 10 hidden neurons
- The training is stopped at error 0.015 after 1418 steps
- $\eta = 0.4$
- The weights are
  $W = [-2.22\ -0.3\ -0.3\ -0.47\ 1.49\ -0.23\ 1.85\ -2.07\ -0.24\ 0.79\ -0.15]^T$,
  $V =$
  $$\begin{bmatrix} 0.57 & 0.66 & -0.1 & -0.53 & 0.14 & 1.06 & -0.64 & -3.51 & -0.03 & 0.01 \\ 0.64 & -0.57 & -1.13 & -0.11 & -0.12 & -0.51 & 2.94 & 0.11 & -0.58 & -0.89 \end{bmatrix}$$
- The result is comparable with previous case
- But more CPU time is required!!.

Initial Weights

- They are usually selected at small random values. (between -1 and 1 or -0.5 and 0.5)
- They affect finding local/global min and speed of convergence
- Choosing them too large saturates network and terminates learning
- Choosing them too small decreases the learning rate.
- They should be chosen s.t do not make the activation function or its derivative zero
- If all weights start with equal values, the network may not train properly.
- Some improper inial weights may result in increasing the errors and decreasing the quality of mapping.
- At these cases the network learning should be restarted with new random weights.

Error

- The training is based on min error
- In delta rule algorithm, Cumulative error is calculated
  $E = \frac{1}{2}\sum_{p=1}^{P}\sum_{k=1}^{R}(d_{pk} - o_{pk})^2$
- Sometimes it is recommended to use $E_{rms} = \frac{1}{pk}\sqrt{(d_{pk} - o_{pk})^2}$
- If output should be discreet (like classification), activation function of output layer is chosen TLU, so the error is

$$E_d = \frac{N_{err}}{pk}$$

  where $Nerr$: # bit errors, p: # training patterns, and k # outputs.
- E_{max} for discreet output can be zero, but in continuous output may not be.

Training versus Generalization

- If learning takes long, network losses the generalization capability. In this case it is said, the network memorizes the training patterns
- To ovoid this problem, Hecht-Nielsen (1990) introduces training-testing pattern (T.T,P)
  - Some specific patterns named T.T.P is applied during training period.
  - If the error obtained by applying the T.T.P is decreasing, the training can be continued.
  - Otherwise, the training is terminated to avoid memorization.

Necessary Number of Patterns for Training set

- Roughly, it can be said that there is a relation between number of patterns, error, and number weights to be trained
- It is reasonable to say number of required pasterns (P) depends
  - directly to # of parameters to be adjusted (weights) (W)
  - inversely to acceptable error (e)
- Beam and Hausler (1989) proposed the following relation

$$P > \frac{32W}{e} ln \frac{32M}{e}$$

  where M is # of hidden layers
- Date Representation
  - For discreet (I/O) pairs it is recommended to use bipolar data rather than binary data
    - Since zero values of input does not contribute in learning
  - For some applications such as identification and control of systems, I/O patterns should be continuous

Necessary Number of Hidden Neurons

- There is no clear and exact rule due to complexity of the network mapping and nondeterministic nature of many successfully completed training procedure.
- # neurons depends on the function to be approximated.
  - Its degree of nonlinearity affects the size of network
- Note that considering large number of neurons and layers may cause overfitting and decrease the generalization capability
- **Number of Hidden Layers**
  - Based on the universal approximation theorem one hidden layer is sufficient for a BP to approximate any continuous mapping from the input patterns to the output patterns to an arbitrary degree of accuracy.
  - More hidden layers may make training easier in some situations or too complicated to converge.

Learning Constant

- Obviously, convergence of error BP alg. depends on the value of η
- In general, optimum value of η depends o the problem to be solved
- When broad minima yields small gradient values, larger η makes the convergence more rapid.
- For steep and narrow minima, small value of η avoids overshooting and oscillation.
- $\therefore$ η should be chosen experimentally for each problem
- Several methods has been introduced to adjust learning const. (η).
- Adaptive Learning Rate in MATLAB adjusts η based on increasing/decreasing error

- η can be defined exponentially,
  - At first steps it is large
  - By increasing number of steps and getting closer to minima it becomes smaller.
- Momentum method
- This method accelerates the convergence of error BP
- Generally, if the training data are not accurate, the weights oscillate and cannot converge to their optimum values
- In momentum method, the speed of BP error convergence is increased without changing η

- In this method, the current weight adjustment confiders a fraction of the most recent weight

$$\triangle w(t) = -\eta \nabla E(t) + \alpha \triangle w(t-1)$$

where α is pos, const. named momentum const.

- The second term is called momentum term
- If the gradients in two consecutive steps have the same sign, the momentum term is pos. and the weight is changes more
- Otherwise, the weights are changed less, but in direction of momentum
- $\therefore$ its direction is corrected

- Start form A'
- Gradient of A' and A" have the same signs
- ∴ the convergence speeds up
- Now start form B'
- Gradient of B' and B" have the different signs
- $\frac{\partial E}{\partial w_2}$ does not point to min
- adding momentum term corrects the direction towards min
- ∴ If the gradient in two consecutive step changes the sign, the learning const. should decrease in those directions (Jacobs 1988)

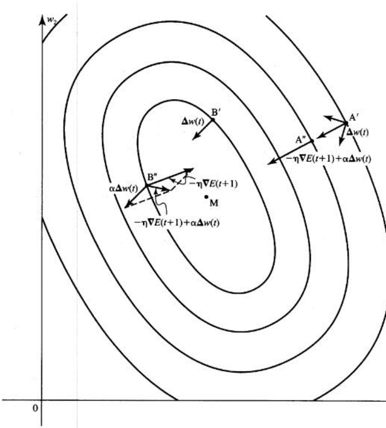


Illustration of adding the momentum term in error back-propagation trainin two-dimensional case.

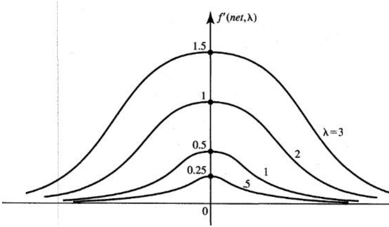
- **Delta-Bar-Delta**
  - For each weight a different η is specified
  - If updating the weight is in the same direction (increasing/decreasing) in some sequential steps, η is increased
  - Otherwise η should decrease
- The updating rule for weight is: $w_{ij}(n+1) = w_{ij}(n) - \eta_{ij}(n+1)\frac{\partial E(n)}{\partial w_{ij}(n)}$
- The learning rate can be updated based on the following rule:

$$\eta_{ij}(n+1) = -\gamma\frac{\partial E(n)}{\partial \eta_{ij}(n)}$$

- where η_{ij} is learning rate corresponding to weights of output layer w_{ij}.
- It can be shown that learning rate is updated based on w_{ij} as follows (**Show it as exercise**) $\eta_{ij}(n+1) = -\gamma\frac{\partial E(n)}{\partial w_{ij}(n)} \cdot \frac{\partial E(n-1)}{\partial w_{ij}(n-1)}$

Steepness of Activation Function

- If we consider $\lambda \neq 1$ is activation function
$$f(net) = \frac{2}{1 + exp(-\lambda net)} - 1$$
- Its time derivative will be $f'(net) = \frac{2\lambda exp(-\lambda net)}{[1+exp(-\lambda net)]^2}$
- max of $f(net)$ when $net = 0$ is $\lambda/2$
- In BP alg: $\triangle w_{ki} = -\eta\delta_{ok}y_j$ where $\delta_{ok} = ef'(net_k)$
- $\therefore$ The weights are adjusted in proportion to $f'(net)$
- slope of $f(net)$ (λ) affects the learning.



- The weights connected to the units responding in their mid-range are changed the most
- The units which are saturated change less.
- In some MLP, the learning constant is fixed and by adapting λ accelerate the error convergence (Rezgui 1991).
- But most commonly, $\lambda = 1$ are fixed and the learning speed is controlled by η

Batch versus Incremental Updates

- Incremental updating: a small weights adjustment follows after each presentation of the training pattern.
  - disadvantage: The network trained this way, may be skewed toward the most recent patterns in the cycle.
- Batch updating: accumulate the weight correction terms for several patterns (or even an entire epoch (presenting all patterns)) and make a single weight adjustment equal to the average of the weight correction terms:

$$\triangle w = \sum_{p=1}^{P} \triangle w_p$$

  - disadvantages: This procedure has a smoothing effect on the correction terms which in some cases, it increases the chances of convergence to a local min.

Normalization

- IF I/O patterns are distributed in a wide range, it is recommended to normalize them before use for training.
- Recall time derivative of sigmoid activation fcn:

$$f'(net) = \begin{cases} o(1-o) & unipolar: \ f(net) = \frac{1}{1+exp(-\lambda net)}, \lambda = 1 \\ \frac{1}{2}(1-o^2) & bipolar: \ f(net) = \frac{2}{1+exp(-\lambda net)} - 1, \lambda = 1 \end{cases}$$

- It appears in δ for updating the weights.
- If output of sigmoid fcn gets to the saturation area, (1 or -1) due to large values of weights or not normalized input data $\rightsquigarrow f'(net) \to 0$ and $\delta \to 0$. So the weight updating is stopped.
- I/O normalization will increases the chance of convergence to the acceptable results.

Offline versus Online Training

- Offline training :
  - After the weights converge to the desired values and learning is terminated, the trained feed forward network is employed
  - When enough data is available for training and no unpredicted behavior is expected from the system, offline training is recommended.
- Online training:
  - Updating the weights and performing the network is simultaneously.
  - In online training NN can adapt it self with unpredicted changing behavior of the system.
  - Learning the the weights convergence should be fast to avoid undesired performance.
  - For exp. if NN is employed as a controller and is not trained fast, it may lead to instability
  - If there is enough data it is suggested to train NN offline and use the trained weight as initial weights in online training to facilitate the training

Levenberg-Marquardt Training [1]

- The LevenbergMarquardt algorithm (LMA) provides a numerical solution to the problem of minimizing a function
- It interpolates between the GaussNewton algorithm (GNA) and gradient descent method.
- The LMA is more robust than the GNA,
  - It will ind the solution even if the initial values are very far off the final minimum.
- In many cases LMA converges faster than gradient decent method.
- LMA is a compromise between the speed of GNA and guaranteed convergence of gradient alg. decent
- Recall the error is defined as sum of squares function for $E = \frac{1}{2}\Sigma_{k=1}^{K}\Sigma_{p=1}^{P}e_{pk}^2$, $e_{pk} = d_{pk} - o_{pk}$
- The learning rule based on gradient decent alg is $\Delta w_{kj} = -\eta\frac{\partial E}{\partial w_{kj}}$

GNA method:

- Define $x = [w^1_{11}\ w^1_{12}\ \ldots w^1_{nm} w^2_{11}\ \ldots w^P_{nm}]$, $e = [e_{11}, \ldots, e_{PK}]$
  - Let Jacobian matrix $J = \begin{bmatrix} \frac{\partial e_{11}}{\partial w^1_{11}} & \frac{\partial e_{11}}{\partial w^1_{12}} & \cdots & \frac{\partial e_{11}}{\partial w^1_{1m}} & \cdots \\ \frac{\partial e_{21}}{\partial w^1_{11}} & \frac{\partial e_{21}}{\partial w^1_{12}} & \cdots & \frac{\partial e_{21}}{\partial w^1_{1m}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial e_{P1}}{\partial w^1_{11}} & \frac{\partial e_{P1}}{\partial w^1_{12}} & \cdots & \frac{\partial e_{P1}}{\partial w^1_{1m}} & \cdots \\ \frac{\partial e_{12}}{\partial w^1_{11}} & \frac{\partial e_{12}}{\partial w^1_{12}} & \cdots & \frac{\partial e_{12}}{\partial w^1_{1m}} & \\ \vdots & \vdots & \vdots & & \end{bmatrix}$
  - and Gradient $\nabla E(x) = J^T(x)e(x)$
  - Hessian Matrix $\nabla^2 E(x) \simeq J^T(x)J(x)$
- Then GNA updating rule is

$$\Delta x = -[\nabla^2 E(x)]^{-1} \nabla E(x) = -[J^T(x)J(x)]^{-1} J^T(x)e$$

Marquardt-Levenberg Alg

$$\Delta x = -[J^T(x)J(x) + \mu I]^{-1} J^T(x)e \tag{3}$$

- μ is a scalar
  - If μ is small, LMA is closed to GNA
  - If μ is large, LMA is closed to gradient decent
- In NN μ is adjusted properly
- for training with LMA, batch update should be applied

Marquardt-Levenberg Training Alg

1. Define an initial values for μ, $\beta > 1$, and E_{max}
2. Present all inputs to the network and compute the corresponding network outputs, and errors. Compute the sum of squares of errors over all inputs E.
3. Compute the Jacobian matrix J
4. Find Δx using (3)
5. Recompute the sum of squares of errors, E using $x + \Delta x$
6. If this new E is larger than that computed in step 2, then increase $\mu = \mu \times \beta$ and go back to step 4.
7. If this new E is smaller than that computed in step 2, then $\mu = \mu / \beta$, let $x = x + \Delta x$,
8. If $E < E_{max}$ stop; otherwise go back to step 2.

M. T. Hagan and M. B. Menhaj.
Training feedforward networks with the marquardt algorithm.
IEEE Trans. on Neural Networks, vol. 5, no 6 , pages 989–993, Nov. 1994.