# Computational Intelligence
# Lecture 13:Fuzzy Logic

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Classical Logic

Fuzzy Logic

The Compositional Rule of Inference

Generalized Modus Ponens

Generalized Modus Tollens

Generalized Hypothetical Syllogism

# Classical Logic

- Logic is the study of methods and principles of reasoning
  - reasoning means obtaining new propositions from existing propositions.
- In classical logic,
  - The propositions are evaluated by true or false.
  - The relationships between propositions are usually expressed by a truth table.
- Logic Formulas: is obtained by combining $-$, $\bigvee$ and $\bigwedge$ in appropriate algebraic expressions
- Tautology: the always true proposition represented by a logic formula, regardless of the truth values of the basic propositions participating in the formula
  - Example: $(p \to q) \leftrightarrow (\bar{p} \vee q)$
- Contradiction: the always false proposition represented by a logic formula, regardless of the truth values of the basic propositions participating in the formula

# Classical Logic

- Inference rules: the forms of tautologies which are used for making deductive inferences
- Some commonly used inference rules are:
  - **Modus Ponens:** $(p \wedge (p \rightarrow q)) \rightarrow q$
    - Premise 1: $x$ is $A$
    - Premise 2:IF $x$ is $A$ THEN $y$ is $B$
    - Conclusion: $y$ is $B$
  - **Modus Tollens:** $(\bar{q} \wedge (p \rightarrow q)) \rightarrow \bar{p}$
    - Premise 1: $y$ is not $B$
    - Premise 2:IF $x$ is $A$ THEN $y$ is $B$
    - Conclusion: $x$ is not $A$
  - **Hypothetical Syllogism:** $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
    - Premise 1: IF $x$ is $A$ THEN $y$ is $B$
    - Premise 2: IF $y$ is $B$ THEN $z$ is $C$
    - Conclusion: IF $x$ is $A$ THEN $z$ is $C$

- **In fuzzy logic**
  - The propositions are fuzzy propositions that are evaluated by memberships between 0 and 1.
  - The ultimate goal is to provide foundations for approximate reasoning with imprecise propositions
- Consider $A, A', B, B'$ are fuzzy sets
- The fundamental principles are
  - **Generalized Modus Ponens**:
    - Premise 1: $x$ is $A'$
    - Premise 2: IF $x$ is $A$ THEN $y$ is $B$
    - Conclusion $y$ is $B'$ s.t. the closer $A'$ to $A$ $\rightsquigarrow$ the closer $B'$ to $B$
    - $A'$ and $B'$ can be

| | $x$ is $A'$ (Premise 1) | $y$ is $B'$ (Conclusion) |
|---|---|---|
| p1 | $x$ is $A$ | $y$ is $B$ |
| p2 | $x$ is very $A$ | $y$ is very $B$ |
| p3 | $x$ is very $A$ | $y$ is $B$ |
| p4 | $x$ is more or less $A$ | $y$ is more or less $B$ |
| p5 | $x$ is more or less $A$ | $y$ is $B$ |
| p6 | $x$ is not $A$ | $y$ is unknown |
| p7 | $x$ is not $A$ | $y$ is not $B$ |

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| p4 | $x$ is more or less $A$ | $y$ is more or less $B$ |
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  - If a causal relation between "$x$ is $A$" and "$y$ is $B$" is not strong in Premise 2, the satisfaction of $p3$ and $p5$ is allowed.
  - $p7$ is based on "IF $x$ is $A$ THEN $y$ is $B$, ELSE $y$ is not $B$."

# Fuzzy Logic

- **Generalized Modus Tollens:**
  - Premise 1: $y$ is $B'$
  - Premise 2: IF $x$ is $A$ THEN $y$ is $B$
  - Conclusion $x$ is $A'$ s.t. the more different $B$ from $B'$ $\rightsquigarrow$ the more different $A$ from $A'$
  - $A'$ and $B'$ can be

| | $y$ is $B'$ (Premise 1) | $x$ is $A'$ (Conclusion) |
|---|---|---|
| t1 | $y$ is $B$ | $x$ is $A$ |
| t2 | $y$ is not very $B$ | $x$ is not very $A$ |
| t3 | $y$ is not more or less $B$ | $x$ is not more or less $A$ |
| t4 | $y$ is not $B$ | $x$ is unknown |
| t5 | $y$ is not $B$ | $x$ is not $A$ |

# Generalized Hypothetical Syllogism:

- Premise 1: IF $x$ is $A$ THEN $y$ is $B$
- Premise 2: IF $y$ is $B'$ THEN $z$ is $C$
- Conclusion: IF $x$ is $A$ THEN $z$ is $C'$ s.t. the closer $B$ to $B'$ $\rightsquigarrow$ the closer $C$ to $C'$
- $A'$ and $B'$ can be

| | $y$ is $B'$ (Premise 1) | $z$ is $C'$ (Conclusion) |
|---|---|---|
| s1 | $y$ is $B$ | $z$ is $C$ |
| s2 | $y$ is very $B$ | $z$ is more or less $C$ |
| s3 | $y$ is very $B$ | $z$ is $C$ |
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- For $s2$
  1. change Premise 1 to IF $x$ is very $A$ THEN $y$ is very $B$
  2. $\therefore$ in Conclusion: IF $x$ is very $A$ THEN $z$ is $C$
  3. To cancel very, use more or less
  4. $\therefore$ IF $x$ is $A$ THEN $z$ is more or less $C$

# Generalized Hypothetical Syllogism:

- Premise 1: IF $x$ is $A$ THEN $y$ is $B$
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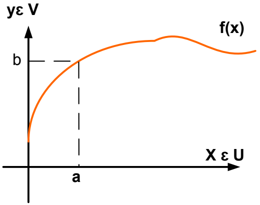
  4. $\therefore$ IF $x$ is $A$ THEN $z$ is more or less $C$
  - The mentioned intuitive criteria are based on approximate reasoning used in daily life They are not necessarily true for classical cases

- How do we determine the membership functions of the fuzzy propositions in the conclusions?
- **The Compositional Rule of Inference**
  - It is a generalization of the following procedure
    - For a curve $y = f(x)$ from $x \in U$ to $y \in V$
    - $x = a$ and $y = f(x) \rightsquigarrow y = b = f(a)$.
    - Now assume $a$ is an interval and $f(x)$ is an interval-valued function
    - First find a cylindrical set $a_E$ with base $a$
    - find $I$: intersection of $A_E$ with the interval-valued curve.
    - The interval $b$: project $I$ on $V$



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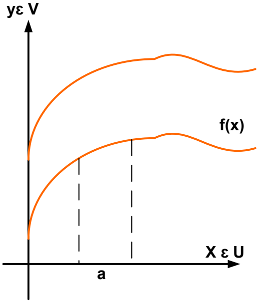

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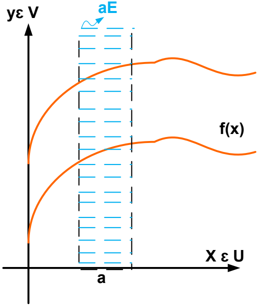

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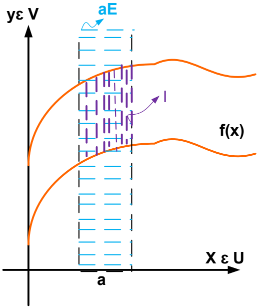

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    - Now assume $a$ is an interval and $f(x)$ is an interval-valued function
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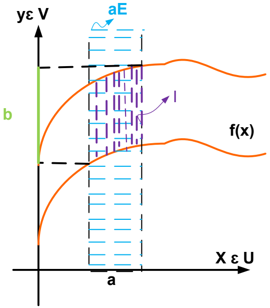

# Compositional Rule of Inference.

- Assume the $A'$ is a fuzzy set in $U$ and $Q$ is a fuzzy relation in $U \times V$.
- Then $A'_E$ is cylindrical extension of $A'$: $\mu_{A'_E}(x, y) = \mu_{A'}(x)$
- $I = A'_E \cap Q \rightsquigarrow \mu_I = t\{\mu_{A'_E}(x, y), \mu_Q(x, y)\} = t\{\mu_{A'}(x), \mu_Q(x, y)\}$
- $B'$ proj. of $I$ on $V$ : $\mu_{B'}(y) = \sup_{x \in U} t\{\mu_{A'}(x), \mu_Q(x, y)\}$
- It is compositional rule of inference.
- **Generalized Modus Ponens:**
  - Fuzzy set $A'$:premise $x$ is $A'$; fuzzy relation $A \rightarrow B \in U \times V$ : premise IF $x$ is $A$ THEN $y$ is $B$; fuzzy set $B' \in V$: conclusion $y$ is $B'$
    $\mu_{B'}(y) = \sup_{x \in U} t[\mu_{A'}(x), \mu_{A \rightarrow B}(x, y)]$
- **Generalized Modus Tollens:**
  - Fuzzy set $B'$:premise $y$ is $B'$; fuzzy relation $A \rightarrow B \in U \times V$ : premise IF $x$ is $A$ THEN $y$ is $B$; fuzzy set $A' \in U$: conclusion $x$ is $A'$
    $\mu_{A'}(x) = \sup_{y \in V} t[\mu_{B'}(y), \mu_{A \rightarrow B}(x, y)]$

- **Generalized Hypothetical Syllogism:**
  - Fuzzy relation $A \to B \in U \times V$ : premise IF $x$ is $A$ THEN $y$ is $B$; Fuzzy relation $B' \to C \in V \times W$ : premise IF $y$ is $B'$ THEN $z$ is $C$; Fuzzy relation $A \to C' \in U \times W$ : conclusion IF $x$ is $A$ THEN $z$ is $C'$; $\mu_{A \to C'}(x, z) = \sup_{y \in V} t[\mu_{A \to B}(x, y), \mu_{B' \to C}(y, z)]$
- Diff. implication principles, definitions of $B', A', C'$ and diff t-norms yields diff. results
- **Generalized Modus Ponens**:
  1. t-norm: min; Mamdani product imp.
     1.1 $A' = A \rightsquigarrow \mu_{B'}(y) = \sup_{x \in U}[\mu_A(x)\mu_B(y)] = \mu_B(y)$
     1.2 $A' =$ very $A \rightsquigarrow \mu_{B'} = \sup_{x \in U}\{\min[\mu_A^2(x), \mu_A(x)\mu_B(y)]\}$ $\sup_{x \in U}\{\mu_A(x)\} = 1$ and $x$ can take any values in $U$, for any $y \in V, \exists x \in U$ s.t. $\mu_A(x) \geq \mu_B(y) \rightsquigarrow \mu_{B'}(y) = \sup_{x \in U}[\mu_A(x)\mu_B(y)] = \mu_B(y)$
     1.3 $A'$ is more or less $A$ $\rightsquigarrow \mu_A^{1/2}(x) \geq \mu_A(x) \geq \mu_A(x)\mu_B(x) \rightsquigarrow \mu_{B'}(y) = \mu_B(y)$
     1.4 $A' = \bar{A}$ for fixed $y \in V, \mu_A(x) \uparrow \rightsquigarrow \mu_A(x)\mu_B(y) \uparrow .1 - \mu_A(x) \downarrow$, $\sup_{x \in U}$ min is obtained when $1 - \mu_A(x) = \mu_A(x)\mu_B(y) \rightsquigarrow \mu_{B'}(y) = \frac{\mu_B(y)}{1+\mu_B(y)}$

- **Generalized Hypothetical Syllogism:**
  - Fuzzy relation $A \to B \in U \times V$ : premise IF $x$ is $A$ THEN $y$ is $B$; Fuzzy relation $B' \to C \in V \times W$ : premise IF $y$ is $B'$ THEN $z$ is $C$; Fuzzy relation $A \to C' \in U \times W$ : conclusion IF $x$ is $A$ THEN $z$ is $C'$; $\mu_{A \to C'}(x,z) = \sup_{y \in V} t[\mu_{A \to B}(x,y), \mu_{B' \to C}(y,z)]$
- Diff. implication principles, definitions of $B', A', C'$ and diff t-norms yields diff. results
- **Generalized Modus Ponens**:
  1. t-norm: min; Mamdani product imp.
     - 1.1 $A' = A \rightsquigarrow \mu_{B'}(y) = \sup_{x \in U}[\mu_A(x)\mu_B(y)] = \mu_B(y)$
     - 1.2 $A' =$ very $A \rightsquigarrow \mu_{B'} = \sup_{x \in U}\{\min[\mu_A^2(x), \mu_A(x)\mu_B(y)]\}$ $\sup_{x \in U}\{\mu_A(x)\} = 1$ and $x$ can take any values in $U$, for any $y \in V, \exists x \in U$ s.t. $\mu_A(x) \geq \mu_B(y) \rightsquigarrow \mu_{B'}(y) = \sup_{x \in U}[\mu_A(x)\mu_B(y)] = \mu_B(y)$
     - 1.3 $A'$ is more or less $A$ $\rightsquigarrow \mu_A^{1/2}(x) \geq \mu_A(x) \geq \mu_A(x)\mu_B(x) \rightsquigarrow \mu_{B'}(y) = \mu_B(y)$
     - 1.4 $A' = \bar{A}$ for fixed $y \in V, \mu_A(x) \uparrow \rightsquigarrow \mu_A(x)\mu_B(y) \uparrow .1 - \mu_A(x) \downarrow$, $\sup_{x \in U}$ min is obtained when $1 - \mu_A(x) = \mu_A(x)\mu_B(y) \rightsquigarrow \mu_{B'}(y) = \frac{\mu_B(y)}{1+\mu_B(y)}$

  p1,p3, and p5 are achieved

# Generalized Modus Ponens

2 t-norm: min; Zadeh imp., Assume $\sup_{x \in U}[\mu_A(x)] = 1$

2.1 $A' = A$ $\mu_{B'}(y) = \sup_{x \in U} \min\{\mu_A(x), \max[\min(\mu_A(x), \mu_B(y)), 1 - \mu_A(x)]\}$

- $\sup_{x \in U}[\mu_A(x)] = 1 \rightsquigarrow \sup_{x \in U}$ *min* is achieved at $x_0 \in U$ when $\mu_A(x_0) = \max[\min(\mu_A(x_0), \mu_B(y)), 1 - \mu_A(x_0)]$
- If $\mu_A(x_0) < \mu_B(y) \rightsquigarrow \mu_A(x_0) = \max[\mu_A(x_0), 1 - \mu_A(x_0)]$, it is true when $\mu_A(x_0) \geq 0.5$, since $\sup_{x \in U}[\mu_A(x)] = 1 \rightsquigarrow \mu_B(y) > \mu_A(x_0) = 1$ impossible!
- $\mu_A(x_0) \geq \mu_B(y) \rightsquigarrow \mu_A(x_0) = \max[\mu_B(y), 1 - \mu_A(x_0)]$, If $\mu_B(y) < 1 - \mu_A(x_0) \rightsquigarrow \mu_A(x_0) = 1 - \mu_A(x_0) \rightsquigarrow \mu_A(x_0) = 0.5$; If $\mu_B(y) \geq 1 - \mu_A(x_0) \rightsquigarrow \mu_A(x_0) = \max[0.5, \mu_B(y)]$
- $\therefore \mu_{B'}(y) = \mu_A(x_0) = \max[0.5, \mu_B(y)]$

# Generalized Modus Ponens

2 t-norm: min; Zadeh imp., Assume $\sup_{x\in U}[\mu_A(x)] = 1$

2.2 $A' =$ very $A$

$\mu_{B'}(y) = \sup_{x\in U} \min\{\mu_A^2(x), \max[\min(\mu_A(x), \mu_B(y)), 1-\mu_A(x)]\}$

- $\sup_{x\in U}[\mu_A(x)] = 1 \rightsquigarrow \sup_{x\in U}$ *min* is achieved at $x_0 \in U$ when $\mu_A^2(x_0) = \max[\min(\mu_A(x_0), \mu_B(y)), 1-\mu_A(x_0)]$
- If $\mu_A(x_0) < \mu_B(y) \rightsquigarrow \mu_A^2(x_0) = \max[\mu_A(x_0), 1-\mu_A(x_0)]$, it is true when $\mu_A(x_0) = 1$, $\rightsquigarrow \mu_B(y) > 1$ impossible!
- $\mu_A(x_0) \geq \mu_B(y) \rightsquigarrow \mu_A^2(x_0) = \max[\mu_B(y), 1-\mu_A(x_0)]$, If $\mu_B(y) < 1-\mu_A(x_0) \rightsquigarrow \mu_A^2(x_0) = 1-\mu_A(x_0) \rightsquigarrow \mu_A(x_0) = \frac{\sqrt{5}-1}{2}, \mu_{B'}(y) = \mu_A^2(x_0) = \frac{3-\sqrt{5}}{2}$; If $\mu_B(y) \geq 1-\mu_A(x_0) \rightsquigarrow \mu_{B'}(y) = \mu_A^2(x_0) = \mu_B(y) \geq \frac{3-\sqrt{5}}{2}$
- $\therefore \mu_{B'}(y) = \mu_A^2(x_0) = \max[\frac{3-\sqrt{5}}{2}, \mu_B(y)]$

# Generalized Modus Ponens

2 t-norm: min; Zadeh imp., Assume $\sup_{x\in U}[\mu_A(x)] = 1$

2.3 $A' =$ more or less $A$

$\mu_{B'}(y) = \sup_{x\in U} \min\{\mu_A^{1/2}(x), \max[\min(\mu_A(x), \mu_B(y)), 1 - \mu_A(x)]\}$

- $\sup_{x\in U}[\mu_A(x)] = 1 \rightsquigarrow \sup_{x\in U}$ *min* is achieved at $x_0 \in U$ when $\mu_A^{1/2}(x_0) = \max[\min(\mu_A(x_0), \mu_B(y)), 1 - \mu_A(x_0)]$
- similar to the previous case If $\mu_A(x_0) < \mu_B(y)$ is impossible!
- $\mu_A(x_0) \geq \mu_B(y) \rightsquigarrow \mu_A^{1/2}(x_0) = \max[\mu_B(y), 1 - \mu_A(x_0)]$,
  If $\mu_B(y) < 1 - \mu_A(x_0) \rightsquigarrow \mu_A^{1/2}(x_0) = 1 - \mu_A(x_0) \rightsquigarrow \mu_A(x_0) = \frac{3-\sqrt{5}}{2}, \mu_{B'}(y) = \mu_A^{1/2}(x_0) = \frac{\sqrt{5}-1}{2}$;
  If $\mu_B(y) \geq 1 - \mu_A(x_0) \rightsquigarrow \mu_{B'}(y) = \mu_A^{1/2}(x_0) = \mu_B(y) \geq \frac{\sqrt{5}-1}{2}$
- $\therefore \mu_{B'}(y) = \mu_A^{1/2}(x_0) = \max[\frac{\sqrt{5}-1}{2}, \mu_B(y)]$

2.4 $A' = \bar{A}$

$\mu_{B'}(y) = \sup_{x\in U} \min\{1 - \mu_A(x), \max[\min(\mu_A(x), \mu_B(y)), 1 - \mu_A(x)]\}$

- $\mu_A(x_0) = 0 \rightsquigarrow 1 - \mu_A(x_0) = 1$ and $\max[\min(\mu_A(x), \mu_B(y)), 1 - \mu_A(x)] = 1$
- $\therefore \mu_{B'}(y) = 1$

p6 is satisfied

# Generalized Modus Tollens

1. t-norm: min; Mamdani product imp.

   1.1 $B' = \bar{B} \rightsquigarrow \mu_{A'}(x) = \sup_{y\in V}[1 - \mu_B(y), \mu_A(x)\mu_B(y)]$
   - $\sup_{y\in V}$ min is at $y_0 \in V$ s.t. $1 - \mu_B(y_0) = \mu_A(x)\mu_B(y_0) \rightsquigarrow \mu_B(y_0) = \frac{1}{1+\mu_A(x)}$
   - $\therefore \mu_{A'}(x) = 1 - \mu_B(y_0) = \frac{\mu_A(x)}{1+\mu_A(x)}$

   1.2 $B' =$ is not very $B \rightsquigarrow \mu_{A'}(x) = \sup_{y\in V}\{\min[1 - \mu_B^2(y), \mu_A(x)\mu_B(y)]\}$
   - $\sup_{y\in V}$ min is at $y_0 \in V$ s.t. $1 - \mu_B^2(y_0) = \mu_A(x)\mu_B(y_0) \rightsquigarrow \mu_B(y_0) = \frac{\sqrt{\mu_A^2(x)+4} - \mu_A(x)}{2}$
   - $\therefore \mu_{A'}(x) = 1 - \mu_B(y_0)\mu_A(x) = \frac{\mu_A(x)\sqrt{\mu_A^2(x)+4} - \mu_A^2(x)}{2}$

   1.3 $B'$ is more or less $B$ $\rightsquigarrow \mu_{A'}(x) = \sup_{y\in V}\{\min[1 - \mu_B^{1/2}(y), \mu_A(x)\mu_B(y)]\}$
   - $\sup_{y\in V}$ min is at $y_0 \in V$ s.t. $1 - \mu_B^{1/2}(y_0) = \mu_A(x)\mu_B(y_0) \rightsquigarrow \mu_B(y_0) = \frac{1+2\mu_A(x) - \sqrt{\mu_A^2(x)+1}}{2\mu_A^2(x)}$
   - $\therefore \mu_{A'}(x) = \mu_A(x)\mu_B(y_0) = \frac{1+2\mu_A(x) - \sqrt{\mu_A^2(x)+1}}{2\mu_A(x)}$

# Generalized Modus Tollens

1. t-norm: min; Mamdani product imp.

   1.4 $B' = B \rightsquigarrow \mu_{A'}(x) = \sup_{y \in V}\{\min[\mu_B(y), \mu_A(x)\mu_B(y)]\} = \sup_{y \in V} \mu_B(y)\mu_A(x) = \mu_A(x)$

   - $\therefore \mu_{A'}(x) = \mu_A(x)$
   - $t1$ is satisfied : $y$ is $B \rightsquigarrow x$ is $A$

# Generalized Hypothetical Syllogism

1. t-norm: min; Mamdani product imp.
   1.1 $B' = B \rightsquigarrow \mu_{A \to C'}(x,z) = \sup_{y \in V}\{\min[\mu_A(x)\mu_B(y), \mu_B(y)\mu_C(z)]\} = (\sup_{y \in V} \mu_B(y)) \min[\mu_A(x), \mu_C(z)]$
      - $\sup_{y \in V}[\mu_B(y)] = 1 \rightsquigarrow \mu_{A \to C'}(x,z) = \min[\mu_A(x), \mu_C(z)]$

   1.2 $B' =$ very $B \rightsquigarrow \mu_{A \to C'}(x,z) = \sup_{y \in V}\{\min[\mu_A(x)\mu_B(y), \mu_B^2(y)\mu_C(z)]\}$
      - If $\mu_A(x) > \mu_C(z) \rightsquigarrow \mu_A(x)\mu_B(y) > \mu_B^2(y)\mu_C(z)$
      - $\therefore \mu_{A \to C'}(x,z) = \sup_{y \in V}[\mu_B^2(y)\mu_C(z)] = \mu_C(z)$
      - If $\mu_A(x) \leq \mu_C(z) \rightsquigarrow \sup_{y \in V}$ min is at $y_0 \in V, \mu_A(x)\mu_B(y_0) = \mu_B^2(y_0)\mu_C(z)$
      - $\therefore \mu_B(y_0) = \frac{\mu_A(x)}{\mu_C(z)} \rightsquigarrow \mu_{A \to C'}(x,z) = \mu_A(x)\mu_B(y_0) = \frac{\mu_A^2(x)}{\mu_C(z)}$
      - $\therefore \mu_{A \to C'}(x,z) = \begin{cases} \mu_C(z) & \text{if } \mu_C(z) < \mu_A(x) \\ \frac{\mu_A^2(x)}{\mu_C(z)} & \text{if } \mu_C(z) \geq \mu_A(x) \end{cases}$

# Generalized Hypothetical Syllogism

1. t-norm: min; Mamdani product imp.
   1.3 $B' =$ more or less $B$
   $\rightsquigarrow \mu_{A\to C'}(x,z) = \sup_{y\in V}\{\min[\mu_A(x)\mu_B(y), \mu_B^{1/2}(y)\mu_C(z)]\}$
   - Using similar method to $B' =$very $B$
   - $\mu_{A\to C'}(x,z) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) < \mu_C(z) \\ \frac{\mu_C^2(z)}{\mu_A(x)} & \text{if } \mu_A(x) \geq \mu_C(z) \end{cases}$

   1.4 $B' = \bar{B}$ $\rightsquigarrow \mu_{A\to C'}(x,z) = \sup_{y\in V}\{\min[\mu_A(x)\mu_B(y), (1-\mu_B(y))\mu_C(z)]\}$
   - $\sup_{y\in V}$ min is achieved at $\mu_B(y_0) = \frac{\mu_C(z)}{\mu_A(x)+\mu_C(z)}$
   - $\therefore \mu_{A\to C'}(x,z) = \frac{\mu_A(x)\mu_C(z)}{\mu_A(x)+\mu_C(z)}$