Computational Intelligence
Lecture 12:Fuzzy Logic

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Fall 2011

Classical Logic

Fuzzy Logic
The Compositional Rule of Inference
Generalized Modus Ponens

Classical Logic

- Logic is the study of methods and principles of reasoning
  - reasoning means obtaining new propositions from existing propositions.
- In classical logic,
  - The propositions are evaluated by true or false.
  - The relationships between propositions are usually expressed by a truth table.
- Logic Formulas: is obtained by combining $-$, $\bigvee$ and $\bigwedge$ in appropriate algebraic expressions
- Tautology: the always true proposition represented by a logic formula, regardless of the truth values of the basic propositions participating in the formula
  - Example: $(p \rightarrow q) \leftrightarrow (\bar{p} \vee q)$
- Contradiction: the always false proposition represented by a logic formula, regardless of the truth values of the basic propositions participating in the formula

Classical Logic

- Inference rules: the forms of tautologies which are used for making deductive inferences
- Some commonly used inference rules are:
  - **Modus Ponens:** $(p \wedge (p \rightarrow q)) \rightarrow q$
    - Premise 1: x is A
    - Premise 2:IF x is A THEN y is B
    - Conclusion: y is B
  - **Modus Tollens:** $(\bar{q} \wedge (p \rightarrow q)) \rightarrow \bar{p}$
    - Premise 1: y is not B
    - Premise 2:IF x is A THEN y is B
    - Conclusion: x is not A
  - **Hypothetical Syllogism:** $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
    - Premise 1: IF x is A THEN y is B
    - Premise 2: IF y is B THEN z is C
    - Conclusion: IF x is A THEN z is C

- **In fuzzy logic**
  - The propositions are fuzzy propositions that are evaluated by memberships between 0 and 1.
  - The ultimate goal is to provide foundations for approximate reasoning with imprecise propositions
- Consider A, A', B, B' are fuzzy sets
- The fundamental principles are
  - **Generalized Modus Ponens**:
    - Premise 1: x is A'
    - Premise 2: IF x is A THEN y is B
    - Conclusion y is B' s.t. the closer A' to $A \rightsquigarrow$ the closer B' to B
    - A' and B' can be

	x is A' (Premise 1)	y is B' (Conclusion)
p1	x is A	y is B
p2	x is very A	y is very B
p3	x is very A	y is B
p4	x is more or less A	y is more or less B
p5	x is more or less A	y is B
p6	x is not A	y is unknown
p7	x is not A	y is not B

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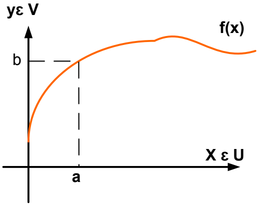
p7	x is not A	y is not B

  - If a causal relation between "x is A" and "y is B" is not strong in Premise 2, the satisfaction of $p3$ and $p5$ is allowed.
  - $p7$ is based on "IF x is A THEN y is B, ELSE y is not B."

- How do we determine the membership functions of the fuzzy propositions in the conclusions?
- **The Compositional Rule of Inference**
  - It is a generalization of the following procedure
    - For a curve $y = f(x)$ from $x \in U$ to $y \in V$
    - $x = a$ and $y = f(x) \rightsquigarrow y = b = f(a)$.
    - Now assume a is an interval and $f(x)$ is an interval-valued function
    - First find a cylindrical set a_E with base a
    - find I: intersection of A_E with the interval-valued curve.
    - The interval b: project I on V



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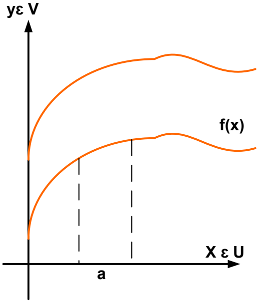

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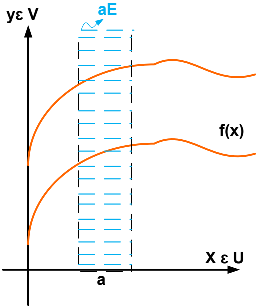

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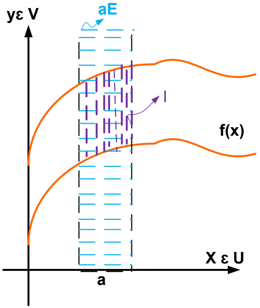

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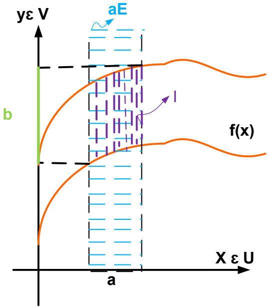

Compositional Rule of Inference.

- Assume the A' is a fuzzy set in U and Q is a fuzzy relation in $U \times V$.
- Then A'_E is cylindrical extension of A': $\mu_{A'_E}(x,y) = \mu_{A'}(x)$
- $I = A'_E \cap Q \rightsquigarrow \mu_I = t\{\mu_{A'_E}(x,y), \mu_Q(x,y)\} = t\{\mu_{A'}(x), \mu_Q(x,y)\}$
- B' proj. of I on V : $\mu_{B'}(y) = \sup_{x \in U} t\{\mu_{A'}(x), \mu_Q(x,y)\}$
- It is compositional rule of inference.
- **Generalized Modus Ponens:**
  - Fuzzy set A':premise x is A'; fuzzy relation $A \rightarrow B \in U \times V$: premise IF x is A THEN y is B; fuzzy set $B' \in V$: conclusion y is B'
    $\mu_{B'}(y) = \sup_{x \in U} t[\mu_{A'}(x), \mu_{A \rightarrow B}(x,y)]$

- Diff. implication principles, definitions of B', A', C' and diff t-norms yields diff. results
- **Generalized Modus Ponens**:
  1. t-norm: min; Mamdani product imp.
     - 1.1 $A' = A \rightsquigarrow \mu_{B'}(y) = \sup_{x\in U}[\mu_A(x)\mu_B(y)] = \mu_B(y)$
     - 1.2 $A' =$ very $A \rightsquigarrow \mu_{B'} = \sup_{x\in U}\{\min[\mu_A^2(x), \mu_A(x)\mu_B(y)]\}$
       $\sup_{x\in U}\{\mu_A(x)\} = 1$ and x can take any values in U, for any $y \in V, \exists x \in U$ s.t.
       $\mu_A(x) \geq \mu_B(y) \rightsquigarrow \mu_{B'}(y) = \sup_{x\in U}[\mu_A(x)\mu_B(y)] = \mu_B(y)$
     - 1.3 (If $\mu_A(x) < \mu_B(y) \Rightarrow \mu_{B'}(y) = \sup \mu_A^2(x) = 1 \rightsquigarrow 1 = \mu_A(x) < \mu_B(y)$ **impossible**)
     - 1.4 A' is more or less A
       $\rightsquigarrow \mu_A^{1/2}(x) \geq \mu_A(x) \geq \mu_A(x)\mu_B(x) \rightsquigarrow \mu_{B'}(y) = \mu_B(y)$
     - 1.5 $A' = \bar{A}$ for fixed $y \in V, \mu_A(x) \uparrow \rightsquigarrow \mu_A(x)\mu_B(y) \uparrow .1 - \mu_A(x) \downarrow$,
       $\sup_{x\in U}$ min is obtained when
       $1 - \mu_A(x) = \mu_A(x)\mu_B(y) \rightsquigarrow \mu_{B'}(y) = \frac{\mu_B(y)}{1+\mu_B(y)}$

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  p1,p3, and p5 are achieved

Generalized Modus Ponens

2 t-norm: min; Zadeh imp., Assume $\sup_{x \in U}[\mu_A(x)] = 1$

2.1 $A' = \bar{A}$

$\mu_{B'}(y) = \sup_{x \in U} \min\{1 - \mu_A(x), \max[\min(\mu_A(x), \mu_B(y)), 1 - \mu_A(x)]\}$

- $\mu_A(x_0) = 0 \rightsquigarrow 1 - \mu_A(x_0) = 1$ and $\max[\min(\mu_A(x), \mu_B(y)), 1 - \mu_A(x)] = 1$
- $\therefore \mu_{B'}(y) = 1$

p6 is satisfied