# Computational Intelligence
# Lecture 11:Fuzzy Relations

**Farzaneh Abdollahi**

Department of Electrical Engineering

Amirkabir University of Technology

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- **Cartesian Product ($U_1 \times U_2 \times \ldots \times U_n$):**
  - $U_i \; t = 1, ..., n$: $n$ arbitrary classical sets.
  - $U_1 \times U_2 \times \ldots \times U_n$ is the set of all ordered $n$-tuples $(u_1, ..., u_n)$:
    $U_1 \times U_2 \times \ldots \times U_n = \{(u_1, u_2, ..., u_n) | u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n\}$
- For binary relation ($n = 2$): $U_1 \times U_2 = \{(u_1, u_2) | u_1 \in U_1, u_2 \in U_2\}$
- $U_1 \neq U_2 \rightsquigarrow U_1 \times U_2 \neq U_2 \times U_1$.
- **A relation among sets $U_1, U_2, \ldots, U_n$ ($Q(U_1, U_2, \ldots, U_n)$)**:
  - a subset of the Cartesian product $U_1 \times U_2 \times \ldots \times U_n$:
    $Q(U_1, U_2, \ldots, U_n) \subset U_1 \times U_2 \times \ldots \times U_n$
- a relation is itself a set $\rightsquigarrow$, all of the basic set operations can be applied to it without modification.
- It can be represented by membership function:

$$\mu_Q(u_1, \ldots, u_n) = \begin{cases} 1 & if \; (u_1, \ldots, u_n) \in Q(U_1, U_2, \ldots, U_n) \\ 0 & otherwise \end{cases}$$

- The values of the membership function $\mu_Q$ can be shown by a relational matrix.

# Example:

- $U = \{1, 2, 3\},\ \ V = \{a, b\}$
- $U \times V = (1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
- Let $Q(U, V)$ be a relation named "the first element is not smaller than 2"
- $Q(U, V) = \{(2, a), (2, b), (3, a), (3, b)\}$

| $U \diagdown V$ | a | b |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 1 | 1 |
| 3 | 1 | 1 |

# Fuzzy Relation

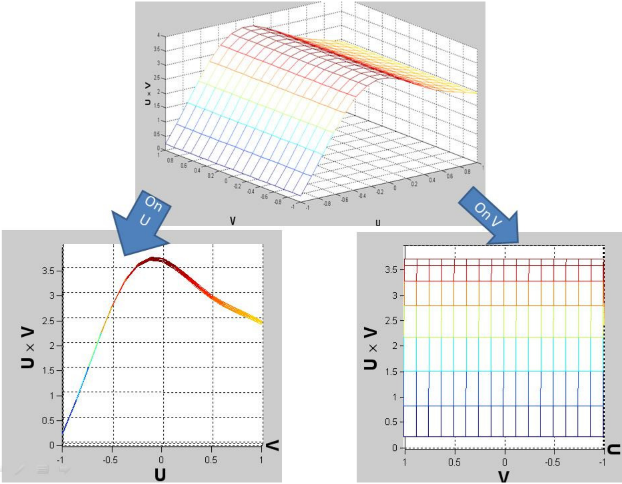
- A classical relation represents a crisp (zero-one) relationship among sets.
- But, for certain relationships, it is difficult to express the relation by a zero-one assessment
- In fuzzy relation the degree the strength of the relation is defined by different membership on the unit interval $[0, 1]$.
- **A fuzzy relation** is a fuzzy set defined in the Cartesian product of crisp sets $U_1, U_2, ..., U_n$.
  $Q = \{((u_1, u_2, ..., u_n), \mu_Q(u_1, u_2, ..., u_n)) | (u_1, u_2, ..., u_n) \in U_1 \times U_2 \times \ldots, \times U_n\}, \quad \mu_Q : U_1 \times U_2 \times \ldots, \times U_n \rightarrow [0, 1]$
- Example:Fuzzy relation: "x is approximately equal to y" (AE).
  - $U = V = R$,
  - $\mu_{AE}(x, y) = e^{-(x-y)^2}$
  - This membership function is not unique

# Example: Dormitory based on Distance of cities.

- $V = \{Tehran, Tabriz, Karaj, Qom\}, U = \{Tehran, Esfahan\}$
- Relation: "very far"
- use number between 0 and 1 for degree of relation

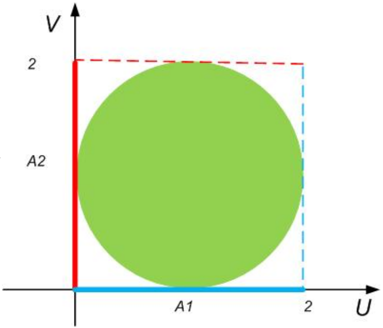
| $U / V$ | Tehran | Tabriz | Karaj | Qom |
|---|---|---|---|---|
| Tehran | 0 | 0.9 | 0.1 | 0.3 |
| Esfahan | 0.7 | 0.95 | 0.8 | 0.5 |

# Projection



# Projection

- **Example:** A crisp relation in $V \times U = R^2$
  - $A = \{(x, y) \in R^2 | (x-1)^2 + (y-1)^2 \leq 1\}$
  - $A_1$ the projection of $A$ on $U$: $[0, 2] \subset U$
  - $A_2$ the projection of $A$ on $V$: $[0, 2] \subset V$

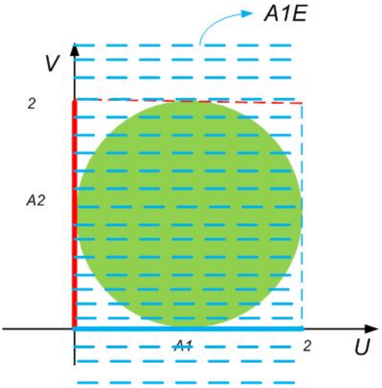


# Projection of Fuzzy Sets

- $Q$: a fuzzy relation in $U_1 \times \ldots \times U_n$
- $\{i_1, \ldots, i_k\}$ a subsequence of $\{1, 2, \ldots, n\}$
- The projection of $Q$ on $U_{i_1} \times \ldots \times U_{i_k}$ is a fuzzy relation $Q_P$ in $U_{i_1} \times \ldots \times U_{i_k}$ s.t.:
  $\mu_{Q_p}(u_{i_1}, \ldots, u_{i_k}) = \underline{\max}_{u_{j_1} \in U_{j_1}, \ldots, u_{j(n-k)} \in U_{j(n-k)}} \mu_Q(u_1, \ldots, u_n)$
- $\{u_{j_1}, \ldots, u_{j(n-k)}\}$ is the complement of $\{u_{i_1}, \ldots, u_{i_k}\}$
- For binary fuzzy relation $U \times V$:
  - $Q_1$ projection in $U$: $\mu_{Q_1}(x) = \max_{y \in V} \mu_Q(x, y)$
- **Example:** Recall the "AE" example
  - Projection in $U$: $AE_1 = \int_U \max_{y \in V} e^{-(x-y)^2}/x = \int_U 1/x$
  - Projection in $V$: $AE_2 = \int_V \max_{x \in U} e^{-(x-y)^2}/y = \int_V 1/y$
- **Example:** Recall the "very far" example
  - $Q_1$: projection on $u$: $Q_1 = 0.9/Tehran + 0.95/Esfahan$
  - $Q_2$: projection on $V$:
    $Q_2 = 0.7/Tehran + 0.95/Tabriz + 0.8/Karaj + 0.5/Qom$

# Cylindrical Extension

- Extending the projection of fuzzy cylindrically.
- **Example:** Recall the circle example
  - $A_{1E}$ Cylindric extention of $A_1$ to $U \times V = R^2$
  - $A_{1E} = [0, 1] \times (-\infty, \infty) \subset R^2$
- The projection constrains a fuzzy relation to a subspace
- The cylindric extension extends a fuzzy relation (or fuzzy set) from a subspace to the whole space.



- Let $Q_p$ be a fuzzy relation in $U_{i_1} \times ... \times U_{i_k}$ and $\{i_1, \ldots, i_k\}$ is a subsequence of $\{1, 2, ..., n\}$, then the cylindric extension of $Q_p$ to $U_l \times \ldots \times U_n$ is a fuzzy relation $Q_{pE}$ in $U_l \times \ldots \times U_n$
  $\mu_{Q_{pE}}(u_1, \ldots, u_n) = \mu_{Q_p}(u_{i_1}, \ldots, u_{i_k})$
- For binary set:
  - $U \times V$,
  - $Q_1$ a fuzzy set in $U$
  - $Q_{1E}$ the cylindric extension to $U \times V$
  - $\mu_{Q_{1E}}(x, y) = \mu_{Q_1}(x)$
- **Example:** Recall the "AE" example
  - $A_{E_{1E}} = \int_{U \times V} 1/(x, y) = U \times V$
  - $A_{E_{2E}} = \int_{U \times V} 1/(x, y) = U \times V$

- **Example:** Recall the "very far" example
  - $Q_{1E}$: cylindrical ext. of $Q_1$ to $U \times V$:
    $Q_{1E} =$
    $0.9/(Tehran, Tehran) + 0.9(Tabriz, Tehran) + 0.9/(Karaj, Tehran) + 0.9/(Qom, Tehran) + 0.95/(Tehran, Esfahan), 0.95/(Tabriz, Esfahan) + 0.95/(Karaj, Esfahan) + 0.95/(Qom, Esfahan)$
  - $Q_{1E}$: cylindrical ext. of $Q_2$ to $U \times V$:
    $Q_{2E} =$
    $0.7/(Tehran, Tehran) + 0.7(Tehran, Esfahn) + 0.95/(Tabriz, Tehran) + 0.95/(Tabriz, Esfahan) + 0.8/(Karaj, Tehran), 0.8/(Karaj, Esfahan) + 0.5/(Qom, Tehran) + 0.5/(Qom, Esfahan)$

# Cartesian Product of Fuzzy Sets

- Let $A_1, \ldots, A_n$ be fuzzy sets in $U_l, \ldots, U_n$, respectively. The Cartesian product of $A_1, \ldots, A_n$ denoted by $A_l \times \ldots \times A_n$, is a fuzzy relation in $U_l \times \ldots \times U_n$:
  $\mu_{A_1 \times \ldots \times A_n}(u_1, \ldots, u_n) = \mu_{A_1}(u_1) * \ldots * \mu_{A_n}(u_n)$
- where $*$ represents any t-norm operator.
- **Lemma:** If $Q$ is a fuzzy relation in $U_l \times \ldots \times U_n$ and $Q_l, \ldots, Q_n$ are its projections on $U_l, \ldots, U_n$, respectively, then

$$Q \subset Q_1 \times \ldots \times Q_n$$

where we use "min" for the t-norm

# Cartesian Product of Fuzzy Sets

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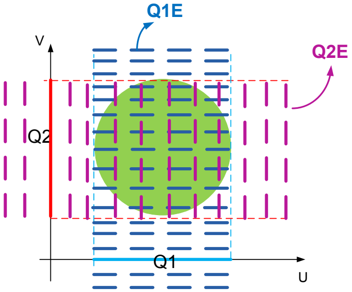
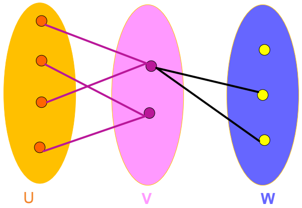


# Composition

- $P(U,V)$ and $Q(V,W)$: two crisp binary relations that share a common set $V$.
- The composition of $P$ and $Q$, $(PoQ)$, is a relation in $U \times W$ s.t. $(x,z) \in Q$ iff there exists at least one $y \in V$ s.t. $(x,y) \in P$ and $(y,z) \in Q$.
- **Lemma**: *$PoQ$ is the composition of $P(U,V)$ and $Q(V,W)$ iff*

$$\mu_{PoQ}(x,z) = \max t[\mu_P(x,y), \mu_Q(y,z)] \quad (1)$$

for any $(x,z) \in U \times W$, where t is any t-norm.



- Proof:
  - If $PoQ$ is the composition:
    - $(x, z) \in PoQ \rightsquigarrow \exists y \in V$ s.t. $\mu_P(x, y) = 1 \& \mu_Q(y, z) = 1$
    - $\therefore \mu_{PoQ}(x, z) = 1 = \max_{y \in V} t[\mu_P(x, y), \mu_Q(y, z)]$
    - If $(x, z) \notin PoQ \rightsquigarrow$ for any $y \in V$, $\mu_P(x, y) = 0$ or $\mu_Q(y, z) = 0$
    - $\therefore \mu_{PoQ}(x, z) = 0 = max_{y \in V} t[\mu_P(x, y), \mu_Q(y, z)]$.
    - Eq. (1) is true.
  - Conversely, if the Eq. (1) is true:
    - $(x, z) \in PoQ \rightsquigarrow \max_{y \in V} t[\mu_P(x, y), \mu_Q(y, z)] = 1$
    - $\therefore$ there exists at least one $y \in V$ s.t. $\mu_P(x, y) = \mu_Q(Y, z) = 1$ ( Axiom t1)
    - For $(x, z) \notin PoQ \rightsquigarrow \max_{y \in V}[\mu_P(x, y), \mu_Q(y, z)] = 0$
    - $\therefore \nexists y \in V$ s.t. $\mu_P(x, y) = \mu_Q(y, z) = 1$.
    - $\therefore PoQ$ is the composition

# Fuzzy Composition

- Composition for fuzzy relations is defined similar to crisp relations
- Based on different definition of t-norm different composition is obtained.
- The two most popular compositions:
  - Max-Min: of fuzzy relations $P(U,V)$ and $Q(V,W)$ is a fuzzy relation $PoQ$ in $U \times W$ s.t. $\mu_{PoQ}(x,z) = \max_{y \in V} \min[\mu_P(x,y), \mu_Q(y,z)]$
    - It uses the min for t-norm

    where $(x,z) \in U \times W$.
  - Max-Product: of fuzzy relations $P(U,V)$ and $Q(V,W)$ is a fuzzy relation $PoQ$ in $U \times W$ s.t. $\mu_{PoQ}(x,z) = \max_{y \in V}[\mu_P(x,y).\mu_Q(y,z)]$ where $(x,z) \in U \times W$.
    - It uses algebraic product for t-norm

# Example: Recall Dormitory example

- $V = \{Tehran, Tabriz, Karaj, Qom\}$, $U = \{Tehran, Esfahan\}$, $W = \{Boomehen, Kashan, Ardebil\}$
- $P(U, V)$ "very far"

| $U / V$ | Tehran | Tabriz | Karaj | Qom |
|---|---|---|---|---|
| Tehran | 0 | 0.9 | 0.1 | 0.3 |
| Esfahan | 0.7 | 0.95 | 0.8 | 0.5 |

# Example: Recall Dormitory example

- $V = \{Tehran, Tabriz, Karaj, Qom\}$, $U = \{Tehran, Esfahan\}$, $W = \{Boomehen, Kashan, Ardebil\}$
- $P(U, V)$ "very far"

| $U \diagup V$ | Tehran | Tabriz | Karaj | Qom |
|---|---|---|---|---|
| Tehran | 0 | 0.9 | 0.1 | 0.3 |
| Esfahan | 0.7 | 0.95 | 0.8 | 0.5 |

- $Q(V, W)$:"very near"

| $V \diagup W$ | Boomehen | Kashan | Ardebil |
|---|---|---|---|
| Tehran | 1 | 0.4 | 0.1 |
| Tabriz | 0.2 | 0 | 0.8 |
| Karaj | 0.6 | 0.3 | 0.1 |
| Qom | 0.7 | 0.95 | 0 |

- $PoQ(U, W)$ using Max-min

**P**

| $U / V$ | Tehran | Tabriz | Karaj | Qom |
|---|---|---|---|---|
| Tehran | 0 | 0.9 | 0.1 | 0.3 |
| Esfahan | 0.7 | 0.95 | 0.8 | 0.5 |

**Q**

| $V \setminus W$ | Boomehen | Kashan | Ardebil |
|---|---|---|---|
| Tehran | 1 | 0.4 | 0.1 |
| Tabriz | 0.2 | 0 | 0.8 |
| Karaj | 0.6 | 0.3 | 0.1 |
| Qom | 0.7 | 0.95 | 0 |

**Min**

| | | | |
|---|---|---|---|
| **0** | **0.9** | **0.1** | **0.3** |
| **1** | **0.2** | **0.6** | **0.7** |
| **0** | **0.2** | **0.1** | **0.3** |

**Max**

**PoQ**

| $U / W$ | Boomehen | Kashan | Ardebil |
|---|---|---|---|
| Tehran | 0.3 | 0.3 | 0.8 |
| Esfahan | 0.7 | 0.5 | 0.8 |

- $PoQ(U, W)$ using Max-Product

**P**

| $U \diagup V$ | Tehran | Tabriz | Karaj | Qom |
|---|---|---|---|---|
| Tehran | 0 | 0.9 | 0.1 | 0.3 |
| Esfahan | 0.7 | 0.95 | 0.8 | 0.5 |

**Q**

| $V \diagdown W$ | Boomehen | Kashan | Ardebil |
|---|---|---|---|
| Tehran | 1 | 0.4 | 0.1 |
| Tabriz | 0.2 | 0 | 0.8 |
| Karaj | 0.6 | 0.3 | 0.1 |
| Qom | 0.7 | 0.95 | 0 |

| | | | | |
|---|---|---|---|---|
| | **0** | **0.9** | **0.1** | **0.3** |
| **x** | **1** | **0.2** | **0.6** | **0.7** |
| | **0** | **0.18** | **0.06** | **0.21** |

**Max**

**PoQ**

| $U \diagup W$ | Boomehen | Kashan | Ardebil |
|---|---|---|---|
| Tehran | 0.21 | 0.0.285 | 0.72 |
| Esfahan | 0.7 | 0.475 | 0.76 |

- The relational matrix for the fuzzy composition $PoQ$ can be computed according to the following method:
  - For max-min composition
    - write out each element in the matrix product $PQ$, But treat:
    - each multiplication as a min operation
    - each addition as a max operation
  - For max-product composition,
    - write out each element in the matrix product $PQ$, but treat
    - each addition as a max operation.

# Extension Principle

- Objective: the domain of a function be extended from crisp points in $U$ to fuzzy sets in $U$
- $f : U \rightarrow V$ a function from crisp set $U$ to crisp set $V$.
- $A$: a fuzzy set $U$
- $B = f(A)$ a fuzzy set in $V$
  - If $f$ is an one-to-one mapping
    $\mu_B(y) = \mu_A[f^{-1}(y)], y \in V$
  - where $f[f^{-1}(y)] = y$
- If $f$ is not one-by-one what should we do ?:(
- Example: $f(x_1) = f(x_2) = y, \; x_1 \neq x_2 \rightsquigarrow \mu_A(x_1) \neq \mu_A(x_2)$
  - Two different values is obtained for $\mu_B(y)$

# Extension Principle

- Extension Principle: $\mu_B(y) = \max_{x \in f^{-1}(y)} \mu_A(x), y \in V$
  - $f^{-1}(y)$: set of all points $x \in U$ s.t. $f(x) = y$
- **Example** $U = \{1, \ldots, 10\}, x \in U, f(x) = x^2 \in V = \{1, \ldots, 100\}$
- Fuzzy set: " small"$= 1/1 + 1/2 + 0.8/3 + 0.6/4 + 0.4/5$
- $\therefore$ "small$^2 = 1/1 + 1/4 + 0.8/9 + 0.6/16 + 0.4/25$