Computational Intelligence
Lecture 11: Classification

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Classification

Hard c-Means
- How to choose the optimal partitions?

Fuzzy C-Means
- Fuzzy c-Means Algorithm

Classification

- The data can be classified according to similar patterns, attributes, features, and other characteristics.
- Classification, also termed clustering.
- Why fuzzy clustering?
  - The regularities may not be precisely defined for clustering.
  - Using fuzzy models to formulated problem may be easier to solve computationally.
    - In fuzzy models the variables are continuous $\rightsquigarrow$ their derivatives can be used to find the direction of search. But in crisp model the only can have 0/1 mem values.
- In this lecture we are introducing one of the most famous classification alg.: c-means clustering [1]

Hard c-Means

- Consider:
  - $X = \{x_1, x_2, ..., x_n\}$ set of data
  - c: # of clusters $(1 < c < n)$
    - $c = n$ classes $\rightsquigarrow$ each data sample into its own class
    - $c = 1$ places all data samples into the same class
    - Neither case requires any effort in classification
- a family of sets $\{A_i, i = 1, 2, ..., c\}$ as a hard c-partition of X if
  - $\bigcup_{i=1}^{c} A_i = X$
  - $A_i \cap A_j = \emptyset \quad 1 \leq i \neq j \leq c$
  - $0 \subset A_i \subset X$

Hard c-Means

- c-partition can be reformulated by mem. fcn ($\mu_{A_i}(x_k) = \mu_{ik}$).
  - $\mu_{ik} = \begin{cases} 1 & x_k \in A_i \\ 0 & x_k \neq A_i \end{cases} \quad k = 1, ..., n, \ \ i = 1, ..., c, \ \ A_i \subset X, x_k \in X$
  - Given the value of μ_{ik}, hard c-partitions X uniquely, and vice versa.
- The μ_{ik}'s should satisfy the following three conditions
  - $\mu_{ik} \in \{0, 1\}, \ 1 \leq i \leq c, 1 \leq k \leq n$
  - $\sum_{i=1}^{c} \mu_{ik} = 1, \ \forall k \in \{1, ..., k\}$
  - $0 < \sum_{k=1}^{n} \mu_{ik} < n, \forall i \in \{1, ..., c\}$

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  - $0 < \sum_{k=1}^{n} \mu_{ik} < n, \forall i \in \{1, ..., c\}$
- The first two conditions $\rightsquigarrow$ each $x_k \in X$ should belong to one and only one cluster.
- The last condition $\rightsquigarrow$ each cluster A_i must contain at least one and at most $n - 1$ data points.
- μ_{ik} can be shown in a $U_{c\times n}$ matrix

Hard c-Means

- Example: Let $X = \{x_1 = orange, x_2 = apple, x_3 = cucumber\}$
- For $c = 2$, which of the following matrices can be hard c-partition matrix?
$$U_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, U_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$
$$U_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, U_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
- Hard c-partition space for X (M_c) is a set of all possible hard c-partition matrices U

How to choose the optimal partitions?

- For example, $c = 10$ and $n = 25$, there are roughly 10^{18} distinct 10-partitions!!
- There are three types of methods for finding optimal partitions:
  1. Hierarchical methods
     - merging and splitting to construct new clusters are based on some measure of similarity
     - The result is a hierarchy of nested clusters.
  2. Graph-theoretic methods
     - x_i are considered as nodes which are connected to each other through edges
     - The criterion for clustering is typically some measure of connectivity
  3. Objective function methods
     - an objective function measuring the "desirability" of clustering candidates is established
     - local minima of the objective function are defined as optimal clusters.

- c-means method is an objective fun. method
- The most popular objective function: overall within-group sum of squared errors:
  $J_w(U,V) = \sum_{k=1}^{n} \sum_{i=1}^{c} \mu_{ik} \|x_k - v_i\|^2$
  - $U = [\mu_{ik}]$
  - $V = \{v_1, ..., v_c\}$, v_i: center of cluster A_i $v_i = \frac{\sum_{k=1}^{n} \mu_{ik} x_k}{\sum_{k=1}^{n} \mu_{ik}}$
  - v_i is the average of all the points in cluster A_i
  - The closer the points of each cluster to their centers (v_i), the smaller the $J(U,V)$

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  - The closer the points of each cluster to their centers (v_i), the smaller the $J(U,V)$
- **How to find the optimal pair (U,V) for J_w?**
- c-means alg. (ISODATA alg.) can provide an optimal method
- The objective function is developed to achieve two goals simultaneously:
  1. Minimize the Euclidean distance between each data point in a cluster and its cluster center
  2. Maximize the Euclidean distance between cluster centers.

Hard c-Means Algorithm

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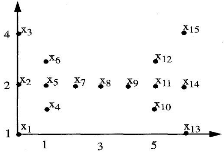
Example

- Suppose $n = 15$, $c = 2$
- $U^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$
- The hard c-means algorithm stops at $l = 3$ with $U^{(3)} = U^{(4)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$
- $\therefore x_1$ to x_7 are grouped into A_1
- x_8 to x_{15} are grouped into A_2
- Although the fig is symmetric the clusters are unsymmetry
- x_8 cold not belong to both clusters
- A way to solve this problem is using the fuzzy c-means alg.



Fuzzy C-Means

- in fuzzy c-means the mem. fcn (μ_{ik} should respect the following conditions
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- A_i have 1 o n members (last condition of mem. fcn. for hard c-means is relaxed for fuzzy c-means)
- Fuzzy c-partition space for X (M_{fc}) is a set of all possible fuzzy c-partition matrices U

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- Fuzzy c-partition space for X (M_{fc}) is a set of all possible fuzzy c-partition matrices U
- An example of possible U for $n = 3, c = 2$: $U = \begin{bmatrix} 0.9 & 0.2 & 0.9 \\ 0.1 & 0.8 & 0.1 \end{bmatrix}$
- For c-means alg. we are looking $U = [\mu_{ik}] \in M_{fc}$ and $V = [v_1, ..., v_n]$ s.t.
  $J_m(U, V) = \sum_{k=1}^{n} \sum_{i=1}^{c} (\mu_{ik})^m \|x_k - v_i\|^2$ is minimized. ($m \in (1, \infty)$)

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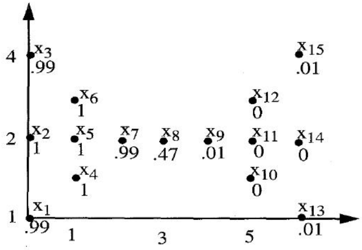
Example

- Consider previous example with $c = 2, m = 1.25, \epsilon = 0.01$
- $U^{(0)} = \begin{bmatrix} 0.854 & 0.146 & 0.854 & 0.854 & \dots & 0.854 \\ 0.146 & 0.854 & 0.146 & 0.146 & \dots & 0.146 \end{bmatrix}$
- The fuzzy c-means algorithm stops at $l = 5$
- The data in the right and left wings are well classified, while the bridge x_8 belongs to both clusters to almost the same degree



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