Nonlinear Control
Lecture 10: Sliding Control

Farzaneh Abdollahi

Department of Electrical Engineering

Amirkabir University of Technology

Fall 2009

Sliding Control

- Sliding Surface
- Integral Control
- Gain Margins

Continuous Approximations of Switching Control Laws

Sliding Control

- Sliding control is a robust control technique to control systems with model imprecision and uncertainties.
- Sliding control is based on the idea that "controlling a 1^{st} order system is much easier than the general n^{th} order system
- To achieve this goal:
  1. A first order system (sliding surface) is proposed and provide a condition (sliding condition) to make the introduced surface an invariant set of the system stability of the system
  2. A control is designed to *reach* to the sliding surface
- Providing perfect performance in presence of arbitrary parameter inaccuracy is at the price of extremely high control activity.
- $\therefore$ a modification of control law is required to provide an effective trade-off between tracking performance and parametric uncertainty.
- In some specific applications, such as those involving the control of electric motor the unmodified control law can be applied directly.

Sliding Surface

- Consider single input dynamics

$$x^{(n)} = f(x) + b(x)u \tag{1}$$

  - f is not exactly known, upper bounded by known continuous function of x
  - b is not exactly known, its sign is known and upper bounded by known continuous function of x
- Objective: find u, s.t. $\mathbf{x}$ track $\mathbf{x}_d = [x_d, \dot{x}_d, \ldots, x_d^{(n-1)}]^T$ in presence of imprecision on $f(x)$ and $b(x)$
- Tracking error vector: $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_d = [\tilde{x}\ \dot{\tilde{x}} \ldots \tilde{x}^{(n-1)}]$
- Define a time-varying surface $S(t)$ in state-space R^n by scaler equation $s(\mathbf{x}; t) = 0$:

$$s(\mathbf{x}; t) = (\frac{d}{dt} + \lambda)^{n-1}\tilde{x} \tag{2}$$

  where $\lambda > 0$ conts..
  - for $n = 2 \rightsquigarrow s = \dot{\tilde{x}} + \lambda\tilde{x}$, s is a weighted sum of position error and velocity error
  - for $n = 3 \rightsquigarrow s = \ddot{\tilde{x}} + 2\lambda\dot{\tilde{x}} + \lambda^2\tilde{x}$

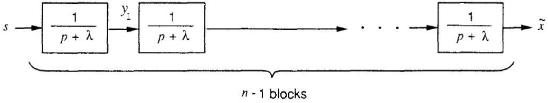
Sliding Surface

- **The problem of tracking the n-dimensional vector x_d (the original tracking problem) can be replaced by a 1st-order stabilization problem in s.**
  - Given initial condition $\mathbf{x}_d(0) = \mathbf{x}(0)$, the problem of tracking $x \equiv x_d$ is equivalent to remaining on the surface $S(t)$ for all $t > 0$ ($s \equiv 0$ represents a linear differential equation whose unique solution is $\tilde{x} \equiv 0$)
  - In (1),s contains $\tilde{x}(n-1) \rightsquigarrow$, we only need to differentiate s once for the input u to appear.
  - Bounds on s can be directly translated into bounds on $\tilde{x} \rightsquigarrow s$ represents a true measure of tracking performance. When $\tilde{\mathbf{x}} = 0$:

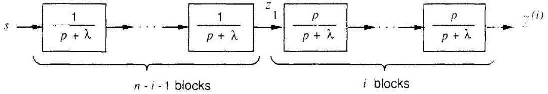
$$\forall t \geq 0, |s(t)| \leq \Phi \Rightarrow \forall t \geq 0, |\tilde{x}^{(i)}| \leq (2\lambda)^i \varepsilon, \quad i = 0, ..., n-1 \tag{3}$$

where $\varepsilon = \Phi/\lambda^{n-1}$

- Proof: $\tilde{x}$ is obtained from s through a sequence of first-order lowpass filters, shown in Fig.



- Let y_1 output of first filter: $y_1 = \int_0^t e^{-\lambda(t-T)} s(T)dT, |s| \leq \Phi \Rightarrow$ $|y_1| \leq \Phi \int_0^t e^{-\lambda(t-T)} dT = (\Phi/\lambda)(1 - e^{-\lambda t}) \leq \Phi/\lambda$
- Repeat the same procedure all the way to $y_{n-1} = \tilde{x} \rightsquigarrow |\tilde{x}| \leq \Phi/\lambda^{n-1} = \varepsilon$
- To obtain $\tilde{x}^{(i)}$, see the Fig b



- The output of the $(n-1-i)^{th}$ filter: $z_1 < \Phi/\lambda^{n-1-i}$
- Note that $\frac{p \pm \lambda}{p+\lambda} = 1 - \frac{\lambda}{\lambda+p} \leq 1 + \frac{\lambda}{\lambda+p}$
- $\therefore |\tilde{x}^{(i)}| \leq (\Phi/\lambda^{n-1-i})(1 + \frac{\lambda}{\lambda})^i = (2\lambda)^i \varepsilon$
- If $\tilde{\mathbf{x}} \neq 0$, $\rightsquigarrow$, (3) is obtained asymptotically, within a short time-constant $(n-l)/\lambda$.

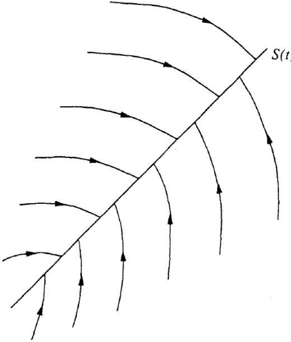
Sliding Condition

- To keep the scalar s at zero, a control law u in should be found s.t outside of $S(t)$:

$$\frac{1}{2}\frac{d}{dt}s^2 \leq -\eta|s| \qquad (4)$$

where $\eta > 0$ conts.

- $\therefore$ The squared "distance" to the surface, s^2 , decreases along all system trajectories. $(V = \frac{1}{2}s^2)$
- (4), so-called sliding condition, makes the surface an *invariant set*.
- By keeping the invariant set, some disturbances or dynamic uncertainties can be tolerated.
- $S(t)$ is sliding surface; behavior of the system on the surface is sliding mode

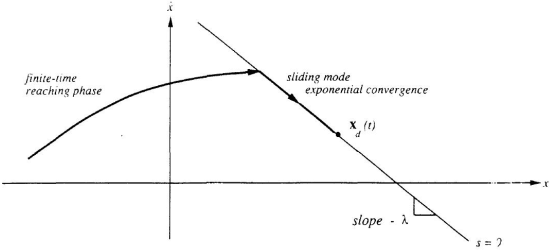


The sliding condition

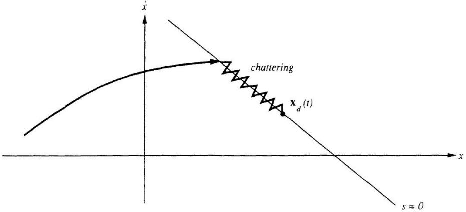
- If it is on the sliding surface, the system behavior can be expressed by $(\frac{d}{dt} + \lambda)^{n-1}\tilde{x} = 0$
- If sliding condition is guaranteed, for *nonzero* initial condition, $(\mathbf{x}(0) \neq \mathbf{x}_d(0))$, the surface $S(t)$ will be reached in a finite time smaller than $|s(t=0)|/\eta$:
  - For t_{reach}: required time to reach $s = 0$, integrate (4) from 0 to t_{reach}: $s(t_{reach}) - s(0) = 0 - s(0) < -\eta(t_{reach} - 0) \rightsquigarrow t_{reach} \leq |s(t=0)|/\eta$
- Once on the surface, tracking error tends exponentially to zero with time constant $(n-1)/\lambda$
  - from the sequence of $(n-1)$ filters of time constants equal to $1/\lambda$

Local Asymptotic Stabilization

- For $n = 2$
  - sliding surface is a line with slope $-\lambda$
  - Starting with any initial conditions, the traj. reaches the time-varying surface in finite time $\leq |s(t = 0)|/\eta$
  - Then slide along the surface towards x_d exp. with time constant $1/\lambda$



- After defining the sliding surface s, the control is designed in two steps
  1. A feedback control law u is selected so as to verify sliding condition (4)
  2. The discontinuous control law u is suitably smoothed to achieve an optimal trade-off between control bandwidth and tracking precision
     - To cope with modeling imprecision and disturbances, the control law has to be discontinuous across $S(t)$.
     - Implementing the associated control switchings is always imperfect (switching is not instantaneous, and the value of s is not known with infinite precision) $\rightsquigarrow$, yields chattering
     - chattering $\rightsquigarrow$ high control activity and may excite high frequency dynamics neglected in modeling (such as unmodeled structural modes, neglected time-delays, and so on).



Chattering as a result of imperfect control switchings

Example

- Consider

$$\ddot{x} = f + u \tag{5}$$

- f is unknown,but estimated by $\hat{f}$, estimation error on f assumed to be bounded by known function $F = F(x, \dot{t})$ $|\hat{f} - f| \leq F$
- To track $x \equiv x_d$, define the sliding surface:

$$s = (\frac{d}{dt} + \lambda)\tilde{x} = \dot{\tilde{x}} + \lambda\tilde{x} \rightsquigarrow \dot{s} = f + u - \ddot{x}_d + \lambda\dot{\tilde{x}}$$

- Best approximation $\hat{u}$ to achieve $\dot{s} = 0$

$$\hat{u} = -\hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}}$$

- The feedback control strategy is chosen intuitive "*if the error is negative, push hard enough in the positive direction (and conversely)*"
- To satisfy (4), a term discontinuous across the surface $s = 0$:

$$u = \hat{u} - ksgn(s)$$

where
$$sgn(s) = 1 \quad if \;\; s > 0$$
$$sgn(s) = -1 \quad if \;\; s < 0$$

- Note that this strategy works only for first-order systems.
- By choosing k to be large enough (4) can be guaranteed

$$\frac{1}{2}\frac{d}{dt}s^2 = \dot{s}.s = (f - \hat{f})s - k|s|$$

,

- letting $k = F + \eta \rightsquigarrow \frac{1}{2}\frac{d}{dt}s^2 \leq -\eta|s|$
- **Integral Control**: To minimize the reaching time and make $s(t=0)=0$, one can use integral control, i.e. $\int_0^t \tilde{x}(r)dr$ as variable of interest.
  - In the previous example, let us define s as $s = (\frac{d}{dt} + \lambda)^2(\int_0^t \tilde{x}(r)dr) = \dot{\tilde{x}} + 2\lambda\tilde{x} + \lambda^2\int_0^t \tilde{x}(r)dr$
  - The approximation of control law will be changed to

$$\hat{u} = -\hat{f} + \ddot{x}_d - 2\lambda\dot{\tilde{x}} - \lambda^2\tilde{x}$$

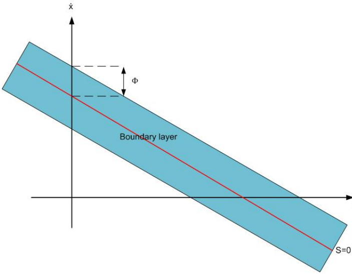
  - The control law, u and k will remain the same
  - Now if $\tilde{x}(0) \neq 0 \rightsquigarrow s = \dot{\tilde{x}} + 2\lambda\tilde{x} + \lambda^2\int_0^t \tilde{x}(r)dr - \dot{\tilde{x}}(0) - 2\lambda\tilde{x}(0)$
  - $\therefore$ Although $\tilde{x}(0) \neq 0$, $s(t=0)=0$

Gain Margins

- Consider
$$\ddot{x} = f + bu$$
where the control gain , b which is may be time-varying or state-dependent is unknown, but of known bounds
$$0 < b_{min} \leq b \leq b_{max}$$
  - choose estimation of b as its geometric mean of bounds: $\hat{b} = (b_{min} b_{max})^{1/2}$.
  - $\therefore \beta^{-1} \leq \frac{\hat{b}}{b} \leq \beta$,
  - $\beta = (b_{max}/b_{min})^{1/2}$ is gain margin
- With s and $\hat{u}$ defined in previous example $u = \hat{b}^{-1}[\hat{u} - ksgn(s)]$
- $\dot{s} = (f - b\hat{b}^{-1}\hat{f}) + (1 - b\hat{b}^{-1})(-\ddot{x}_d + \lambda\dot{\tilde{x}}) - b\hat{b}^{-1}ksgn(s)$
- $\therefore$ to satisfy sliding condition
$k \geq |\hat{b}b^{-1}f - \hat{f} + (\hat{b}b^{-1} - 1)(-\ddot{x}_d + \lambda\dot{\tilde{x}})| + \eta\hat{b}b^{-1}$
- Since $f = \hat{f} + (f - \hat{f})$, where $|f - \hat{f}| \leq F \rightsquigarrow k \geq \beta(F + \eta) + (\beta - 1)|\hat{u}|$

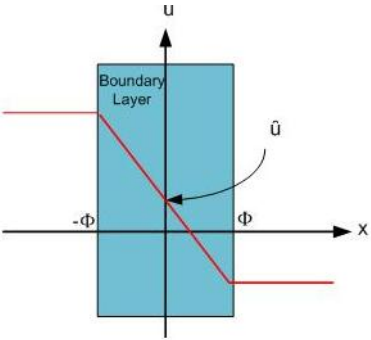
Continuous Approximations of Switching Control Laws

- For system dynamics (1) a unique smooth control to track a feasible trajectory is $u(t) = b(x_d)^{-1}[\ddot{x}_d - f(x_d)]$
- Control laws obtained by using sliding control which provides "perfect" tracking in the face of model uncertainty, are discontinuous across the surface $S(t)$, $\rightsquigarrow$, chattering.
- In general, chattering is not undesirable, since it causes high control activity, and may excite high-frequency dynamics neglected in modeling
- The chattering is avoided by smoothing out the control discontinuity in a thin boundary layer neighboring the switching surface



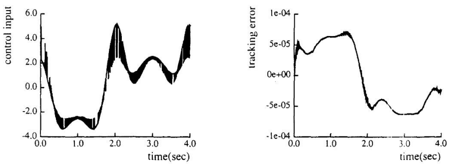
$B(t) = \{X, |s(x;t)| \leq \Phi\}, \quad \Phi > 0$ is the boundary layer thickness

- Outside of $B(t)$, the control law u is like before to guarantee that the boundary layer is invariant
  - All trajectories starting inside $B(t=0)$ remain inside $B(t)$ for all $t > 0$
- Inside $B(t)$, u is interpolated
  - For instance,, inside $B(t)$, in the expression of u replace $sgn(s)$ by s/Φ, as shown in Fig



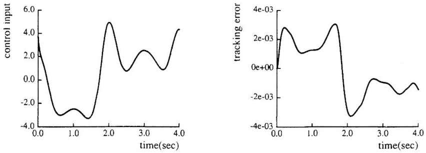
- **Example**: Consider the system dynamics
  $$\ddot{x} + a(t)\dot{x}^2 \cos 3x = u$$
  - $1 \leq a(t) \leq 2$, for simulation $a(t) = |\sin t| + 1$,
  - $\lambda = 20, \ \eta = 0.1$
  - $\hat{f} = 1.5\dot{x}^2 \cos 3x, \quad F = 0.5\dot{x}^2 |\cos 3x|$
  - By using the switching control law: $u = \hat{u} - ksgn(s) = 1.5\dot{x}^2 cos3x + \ddot{x}_d - 20\dot{\tilde{x}} - (0.5\dot{x}^2|\cos 3x| + 0.1)sgn(\dot{\tilde{x}} + 20\tilde{x})$

Example Cont'd



Switched control input and resulting tracking performance

- Tracking performance is excellent at the price of high control chattering
- Modify control law by considering a thin boundary layer of thickness 0.1
- $u = \hat{u} - ksat(s/\Phi) = 1.5\dot{x}^2 cos3x + \ddot{x}_d - 20\dot{\tilde{x}} - (0.5\dot{x}^2|\cos 3x| + 0.1)sat(\dot{\tilde{x}} + 20\tilde{x}/0.1)$



Smooth control input and resulting tracking performance

- The tracking is not as perfect as before but acceptable, instead the control law is smooth

- The smoothing of control discontinuity inside B(t) actually assigns a lowpass filter structure to the local dynamics of the variables to eliminating chattering
- Recognizing this filter-like structure allows us to
  - tune up the control law by selecting λ and Φ properly s.t achieve a trade-off between tracking precision and robustness to unmodeled dynamics.

Thank You