Nonlinear Control
Lecture 1: Introduction

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Motivations

- A system is called **linear** if its behavior set satisfies linear superposition laws: i.e. \( \forall z_1, z_2 \in \mathcal{B} \) and constant \( c \in \mathcal{R} \Rightarrow z_1 + z_2 \in \mathcal{B}, cz_1 \in \mathcal{B} \)
- A **nonlinear system** is simply a system which is not linear.
- Powerful tools founded based on superposition principle make analyzing the linear systems simple.
- All practical systems possess nonlinear dynamics.
- Sometimes it is possible to describe the operation of physical systems by linear model around its operating points
- Linearized system can provide us an approximate behavior of the nonlinear system
- **But** in analyzing the overall system behavior, often linearized model *inadequate* or *inaccurate*.
Linearization is an approximation in the neighborhood of an operating system which it can only predict local behavior of nonlinear system. (No info regarding nonlocal or global behavior of system)

Due to richer dynamics of nonlinear systems comparing to the linear ones, there are some essentially nonlinear phenomena that can take place only in presence of nonlinearity.

Essentially nonlinear phenomena

- **Finite escape time**: The state of linear system goes to infinity as $t \to \infty$; nonlinear system’s state can go to infinity in finite time.
- **Multiple isolated equilibria**: Linear system can have only one isolated equilibrium point which attracts the states irrespective on the initial state; nonlinear system can have more than one isolated equilibrium point, the state may converge to each depending on the initial states.
- **Limit cycle**: There is no robust oscillation in linear systems. To oscillate there should be a pair of eigenvalues on the imaginary axis which due to presence of perturbations it is almost impossible in practice; For nonlinear systems, there are some oscillations named limit cycle with fixed amplitude and frequency.
Essentially nonlinear phenomena

- **Subharmonic, harmonic or almost periodic oscillations:** A stable linear system under a periodic input $\rightarrow$ output with the same frequency; A nonlinear system under a periodic input $\rightarrow$ can oscillate with submultiple or multiple frequency of input or almost-periodic oscillation.

- **Chaos:** A nonlinear system may have a different steady-state behavior which is not equilibrium point, periodic oscillation or almost-periodic oscillation. This chaotic motions exhibit random, despite of deterministic nature of the system.

- **Multiple modes of behavior:** A nonlinear system may exhibit multiple modes of behavior based on type of excitation:
  - an unforced system may have one limit cycle.
  - Periodic excitation may exhibit harmonic, subharmonic, or chaotic behavior based on amplitude and frequency of input.
  - if amplitude or frequency is smoothly changed, it may exhibit discontinuous jump of the modes as well.
- **Linear systems**: can be described by a set of ordinary differential equations and usually the *closed-form expressions* for their solutions are derivable. **Nonlinear systems**: In general this is not possible \(\Rightarrow\) It is desired to make a prediction of system behavior even in absence of closed-form solution. This type of analysis is called *qualitative analysis*.

- Despite of **linear systems**, no tool or methodology in **nonlinear system analysis** is *universally* applicable \(\Rightarrow\) their analysis requires a wide variety of tools and higher level of mathematic knowledge.

- Therefore, stability analysis and stabilizability of such systems and getting familiar with associated control techniques is the basic requirement of graduate studies in control engineering.

- **The aim of this course are**
  - developing a basic understanding of nonlinear control system theory and its applications.
  - introducing tools such as Lyapunov’s method analyze the system stability
  - Presenting techniques such as feedback linearization to control nonlinear systems.
Reference Books

- **Other reference Books**:
## Topics

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At this course we consider dynamical systems modeled by a finite number of coupled first-order ordinary differential equations:

\[ \dot{x} = f(t, x, u) \]  

where \( x = [x_1, \ldots, x_n]^T \): state vector, \( u = [u_1, \ldots, u_p]^T \): input vector, and \( f(.) = [f_1(\cdot), \ldots, f_n(\cdot)]^T \): a vector of nonlinear functions.

Eq. (1) is called state equation.

Another equation named output equation:

\[ y = h(t, x, u) \]  

where \( y = [y_1, \ldots, y_q]^T \): output vector.

Equ (2) is employed for particular interest in analysis such as

- variables which can be measured physically
- variables which are required to behave in a desirable manner

Equs (1) and (2) together are called state-space model.
Most of our analysis are dealing with unforced state equations where \( u \) does not present explicitly in Equ (1):

\[
\dot{x} = f(t, x)
\]

In unforced state equations, input to the system is **NOT** necessarily zero. Input can be a function of time: \( u = \gamma(t) \), a feedback function of state: \( u = \gamma(x) \), or both \( u = \gamma(t, x) \) where is substituted in Equ (1).

**Autonomous or Time-invariant Systems:**

\[
\dot{x} = f(x)
\]  

function of \( f \) does not explicitly depend on \( t \). Autonomous systems are invariant to shift in time origin, i.e. changing \( t \) to \( \tau = t - a \) does not change \( f \). The system which is not autonomous is called **nonautonomous** or **time-varying**.
Equilibrium Point $x = x^*$

- $x^*$ in state space is equilibrium point if whenever the state starts at $x^*$, it will remain at $x^*$ for all future time.
- for autonomous systems (3), the equilibrium points are the real roots of equation: $f(x) = 0$.
- Equilibrium point can be
  - Isolated: There are no other equilibrium points in its vicinity.
  - a continuum of equilibrium points
Employing Newton’s second law of motion, equation of pendulum motion is:

$$ml\ddot{\theta} = -mg \sin \theta - kl\dot{\theta}$$

$l$: length of pendulum rod;  
$m$: mass of pendulum bob;  
$k$: coefficient of friction;  
$\theta$: angle subtended by rod and vertical axis.

To obtain state space model, let $x_1 = \theta$, $x_2 = \dot{\theta}$:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2$$
Pendulum

- To find equilibrium point: $\dot{x}_1 = \dot{x}_2 = 0$

$$
\begin{align*}
0 &= x_2 \\
0 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2
\end{align*}
$$

- The Equilibrium points are at $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$

  - Pendulum has two equilibrium points: $(0, 0)$ and $(\pi, 0)$,
  - Other equilibrium points are repetitions of these two which correspond to number of pendulum full swings before it rests

- Physically we can see that the pendulum rests at $(0, 0)$, but hardly maintain rest at $(\pi, 0)$
Tunnel Diode Circuit

- The tunnel diode is characterized by $i_R = h(v_R)$.
- The energy-storing elements are $C$ and $L$ which assumed are linear and time-invariant $i_C = C \frac{dv_C}{dt}$, $v_L = L \frac{di_L}{dt}$.
- Employing Kirchhoff’s current law: $i_C + i_R - i_L = 0$
- Employing Kirchhoff’s voltage law: $v_C - E + R i_L + v_L = 0$
- for state space model, let $x_1 = v_C$, $x_2 = i_L$ $u = E$ as a constant input:

$$
\dot{x}_1 = \frac{1}{C}[-h(x_1) + x_2]
$$

$$
\dot{x}_2 = \frac{1}{L}[-x_1 - Rx_2 + u]
$$
Tunnel Diode Circuit

▶ To find equilibrium point: \( \dot{x}_1 = \dot{x}_2 = 0 \)

\[
0 = \frac{1}{C}[-h(x_1) + x_2]
\]
\[
0 = \frac{1}{L}[-x_1 - Rx_2 + u]
\]

▶ Equilibrium points depends on \( E \) and \( R \)

\[
x_2 = h(x_1) = \frac{E}{R} - \frac{1}{R}x_1
\]

▶ For certain \( E \) and \( R \), it may have 3 points (\( Q_1, Q_2, Q_3 \)).
▶ if \( E \uparrow \), and same \( R \) \( \Leftrightarrow \) only \( Q_3 \) exists.
▶ if \( E \downarrow \), and same \( R \) \( \Leftrightarrow \) only \( Q_1 \) exists.