Nonlinear Control
Lecture 1: Introduction

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Motivations

- A system is called linear if its behavior set satisfies linear superposition laws: .i.e. $\forall\ z_1,\ z_2 \in \mathcal{B}$ and constant $c \in \mathcal{R} \rightsquigarrow z_1 + z_2 \in \mathcal{B},\ cz_1 \in \mathcal{B}$
- A nonlinear system is simply a system which is not linear.
- Powerful tools founded based on superposition principle make analyzing the linear systems simple.
- All practical systems posses nonlinear dynamics.
- Sometimes it is possible to describe the operation of physical systems by linear model around its operating points
- Linearized system can provide us an approximate behavior of the nonlinear system
- But in analyzing the overall system behavior, often linearized model *inadequate* or *inaccurate*.

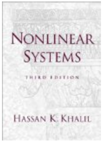
- Linearization is an approximation in the neighborhood of an operating system $\rightsquigarrow$ it can only predict local behavior of nonlinear system. (No info regarding *nonlocal* or *global* behavior of system)
- Due to richer dynamics of nonlinear systems comparing to the linear ones, there are some *essentially nonlinear phenomena* that can take place only in presence of nonlinearity
- Essentially nonlinear phenomena
  - Finite escape time: The state of linear system goes to infinity as $t \to \infty$; nonlinear system's state can go to infinity in *finite time*.
  - Multiple isolated equilibria: linear system can have only one isolated equilibrium point which attracts the states irrespective on the initial state; nonlinear system can have more than one isolated equilibrium point, the state may converge to each depending on the initial states.
  - Limit cycle: There is no robust oscillation in linear systems. To oscillate there should be a pair of eigenvalues on the imaginary axis which due to presence of perturbations it is almost impossible in practice; For nonlinear systems, there are some oscillations named limit cycle with fixed amplitude and frequency.

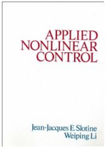
Essentially nonlinear phenomena

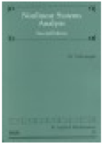
- Subharmonic,harmonic or almost periodic oscillations: A stable linear system under a periodic input $\rightsquigarrow$ output with the same frequency; A nonlinear system under a periodic input $\rightsquigarrow$ can oscillate with submultiple or multiple frequency of input or almost-periodic oscillation.
- Chaos: A nonlinear system may have a different steady-state behavior which is not equilibrium point, periodic oscillation or almost-periodic oscillation. This chaotic motions exhibit random, despite of deterministic nature of the system.
- Multiple modes of behavior: A nonlinear system may exhibit multiple modes of behavior based on type of excitation:
  - an unforced system may have one limit cycle.
  - Periodic excitation may exhibit harmonic, subharmonic,or chaotic behavior based on amplitude and frequency of input.
  - if amplitude or frequency is smoothly changed, it may exhibit discontinuous jump of the modes as well.

- Linear systems: can be described by a set of ordinary differential equations and usually the *closed-form expressions* for their solutions are derivable. Nonlinear systems: In general this is not possible $\rightsquigarrow$ It is desired to make a prediction of system behavior even in absence of closed-form solution. this type of analysis is called qualitative analysis.
- Despite of linear systems, no tool or methodology in nonlinear system analysis is *universally* applicable $\rightsquigarrow$ their analysis requires a wide verity of tools and higher level of mathematic knowledge
- $\therefore$ stability analysis and stabilizablity of such systems and getting familiar with associated control techniques is the basic requirement of graduate studies in control engineering.
- The aim of this course are
  - developing a basic understanding of nonlinear control system theory and its applications.
  - introducing tools such as Lyapunov's method analyze the system stability
  - Presenting techniques such as feedback linearization to control nonlinear systems.

Reference Books







- **Text Book:** Nonlinear Systems, H. K. Khalil, 3^{rd} edition, Prentice-Hall, 2002
- **Other reference Books:**
- Applied Nonlinear Control, J. J. E. Slotine, and W. Li, Prentice-Hall, 1991
- Nonlinear System Analysis, M. Vidyasagar, 2^{nd} edition, Prentice-Hall, 1993
- Nonlinear Control Systems, A. Isidori, 3^{rd} edition Springer-Verlag, 1995

Topics

Topic	Date	Refs
Introduction, Phase plane, Analysis	Weeks 1,2	Chapters 1-3
Stability Theory	Weeks 3-5	Chapters 4,8,9
Input-to-state, I/O stability and Absolute stability	Weeks 6,7	Chapters 4,5
Passivity	Week 8	Chapter 10
Feedback Control	Weeks 9,10	Chapter 12
Feedback linearization	Weeks 11,12	Chapters 12,13
Sliding Mode	Weeks 13,14	
Back Stepping	Weeks 15,16	

- At this course we consider dynamical systems modeled by a finite number of coupled first-order ordinary differential equations:

$$\dot{x} = f(t, x, u) \tag{1}$$

where $x = [x_1, \ldots, x_n]^T$: state vector, $u = [u_1, \ldots, u_p]^T$: input vector, and $f(.) = [f_1(.), \ldots, f_n(.)]^T$: a vector of nonlinear functions.
- Euq. (1) is called *state equation*.
- Another equation named *output equation*:

$$y = h(t, x, u) \tag{2}$$

where $y = [y_1, \ldots, y_q]^T$: output vector.
- Equ (2) is employed for particular interest in analysis such as
  - variables which can be measured physically
  - variables which are required to behave in a desirable manner
- Equs (1) and (2) together are called state-space model.

- Most of our analysis are dealing with unforced state equations where u does not present explicitly in Equ (1):

$$\dot{x} = f(t, x)$$

  - In unforced state equations, input to the system is **NOT** necessarily zero.
  - Input can be a function of time: $u = \gamma(t)$, a feedback function of state: $u = \gamma(x)$, or both $u = \gamma(t, x)$ where is substituted i Equ (1).

- **Autonomous** or **Time-invariant Systems**:

$$\dot{x} = f(x) \tag{3}$$

  - function of f does not explicitly depend on t.
  - Autonomous systems are invariant to shift in time origin, i.e. changing t to $\tau = t - a$ does not change f.
  - The system which is not autonomous is called **nonautonomous** or **time-varying**.

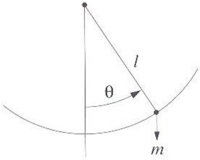
- **Equilibrium Point** $x = x^*$
  - x^* in state space is equilibrium point if whenever the state starts at x^*, it will remain at x^* for all future time.
  - for autonomous systems (3), the equilibrium points are the real roots of equation: $f(x) = 0$.
  - Equilibrium point can be
    - Isolated: There are no other equilibrium points in its vicinity.
    - a continuum of equilibrium points

Pendulum

- Employing Newton's second law of motion, equation of pendulum motion is:

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

l: length of pendulum rod;
m: mass of pendulum bob;
k: coefficient of friction;
θ: angle subtended by rod and vertical a›



Pendulum.

- To obtain state space model,
let $x_1 = \theta$, $x_2 = \dot{\theta}$:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{l}sinx_1 - \frac{k}{m}x_2\end{aligned}$$

Pendulum

- To find equilibrium point: $\dot{x}_1 = \dot{x}_2 = 0$

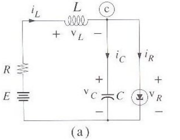
$$\begin{aligned} 0 &= x_2 \\ 0 &= -\frac{g}{l} sin x_1 - \frac{k}{m} x_2 \end{aligned}$$

- The Equilibrium points are at $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$
  - Pendulum has two equilibrium points: $(0, 0)$ and $(\pi, 0)$,
  - Other equilibrium points are repetitions of these two which correspond to number of pendulum full swings before it rests
- Physically we can see that the pendulum rests at $(0, 0)$, but hardly maintain rest at $(\pi, 0)$

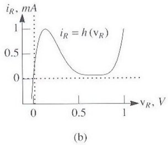
Tunnel Diode Circuit

- The tunnel diode is characterized by $i_R = h(v_R)$
- The energy-storing elements are C and L which assumed are linear and time-invariant $i_C = C\frac{dv_C}{dt}, \ v_L = L\frac{di_L}{dt}$.
- Employing Kirchhoff's current law: $i_C + i_R - i_L = 0$
- Employing Kirchhoff's voltage law: $v_C - E + Ri_L + v_L = 0$
- for state=space model, let $x_1 = v_C, \ x_2 = i$ $u = E$ as a constant input:

$$\dot{x}_1 = \frac{1}{C}[-h(x_1) + x_2]$$
$$\dot{x}_2 = \frac{1}{L}[-x_1 - Rx_2 + u]$$



(a) Tunnel diode circuit



(b) tunnel diode $v_R - i_R$ characteristic.

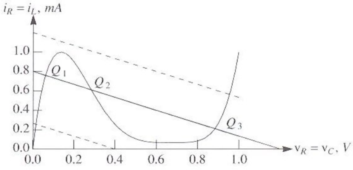
Tunnel Diode Circuit

- To find equilibrium point: $\dot{x}_1 = \dot{x}_2 = 0$

$$0 = \frac{1}{C}[-h(x_1) + x_2]$$
$$0 = \frac{1}{L}[-x_1 - Rx_2 + u]$$

- Equilibrium points depends on E and R

$$x_2 = h(x_1) = \frac{E}{R} - \frac{1}{R}x_1$$



Equilibrium points of the tunnel diode circuit.

- For certain E and R, it may have 3 points (Q_1, Q_2, Q_3).
- if $E\uparrow$, and same $R \rightsquigarrow$ only Q_3 exists.
- if $E\downarrow$, and same $R \rightsquigarrow$ only Q_1 exists.